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QUANTUM ARGUMENT SHIFT METHOD AND QUANTUM MISHCHENKO–FOMENKO ALGEBRAS IN $Ugl(d, \mathbb{C})$

Scientific specialty 1.1.3. Geometry and topology

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Introduction

General Description of the Work

Let M be a Poisson manifold, i.e., a manifold with a fixed Poisson structure on the algebra of smooth functions. Recall that an equation of the form

$$\dot{x} = \{H, x\}$$

is called a Hamiltonian integrable system on the Poisson manifold M, where H is the Hamiltonian function (the energy of the system). The integrability of such systems is understood in the sense of the Liouville theorem, i.e., as the existence of a large system of first integrals in involution, meaning functions $H_1 = H, \dots, H_N$ such that $\{H_i, H_j\} = 0$. One of the main questions in the theory of Hamiltonian integrable systems is the problem of constructing systems of first integrals in involution. An important and effective method for constructing commutative subalgebras in integrable systems is the *argument shift method*, which consists of the fact that, under certain conditions, iterated derivatives (shifts) of functions that are central with respect to the Poisson bracket along a certain vector field commute with each other. We will refer to such a vector field as the shift operator.

On the other hand, with each Poisson manifold, one associates an associative noncommutative multiplication on the space of formal power series $C^\infty(M)[[\hbar]]$, known as the deformation quantization of the manifold M; if $M = g^*$ is the dual space of a Lie algebra g, then the deformation quantization of the manifold M is closely related to the universal enveloping algebra Ug. If $C \in C^{\infty}(M)$ is a Poisson commutative subalgebra (the algebra of integrals of some integrable Hamiltonian system), then the choice of a commutative subalgebra $\hat{C}$ in $C^{\infty}(M)[[\hbar]]$, which "extends" the algebra C, is called a quantum integrable system corresponding to the classical system C. The connection between quantum and classical integrable systems is an important subject of study.

The dissertation is devoted to a question that lies at the intersection of the theory of integrable systems, the theory of groups and Lie algebras, and the theory of deformation quantization — the question of the possibility of transferring the argument shift method to universal enveloping algebras (more generally, to any algebras obtained by deformation quantization from function algebras on Poisson manifolds). This method allows the construction of integrable systems on the dual spaces of Lie algebras. More specifically, the problem studied in the dissertation is the "quantization" (lifting into the universal enveloping algebra Ug) of the argument shift operator on the symmetric algebra Sg in the special case of $g = gl(d, \mathbb{C})$ and the investigation of the properties of the constructed operator.

The work was prepared at the department of differential geometry and applications of the faculty of mechanics and mathematics at Lomonosov Moscow State University. The dissertation is dedicated to one of the important areas of deformation quantizations — the quantum argument shift method. The main problem studied in the dissertation is the quantization of the argument shift operator.

Relevance of the Research Topic

As mentioned above, the argument shift method is one of the effective methods for constructing conserved quantities in integrable systems. The commutative

Poisson subalgebras in the symmetric algebra Sg constructed using this method are called argument shift algebras. The argument shift algebras and their quantization (their "lifting" into the universal enveloping algebra) are subjects of study in many modern works. The question of the existence of operators that lift the shift operator into universal enveloping algebras has not, to our knowledge, been previously discussed in the literature. The presence of such an operator in the theory should help to solve the problem of quantizing (lifting into universal enveloping algebras) other commutative Poisson subalgebras in the symmetric algebra Sg and in other important examples. Ideally, it should help address the question of quantizing the method of bi-Hamiltonian induction (another widely used method for constructing commutative families of functions on Poisson manifolds).

Thus, the main goals of the work can be formulated in two closely related points:

1. To propose a definition of the "quantum" argument shift operator, whose action on the central elements of the universal enveloping algebra would generate a quantum argument shift algebra;

2. To describe the quantum argument shift algebra and its elements using the quantum argument shift operator.

The dissertation provides a solution to these problems for the canonical Poisson structure on the symmetric algebra of the general linear Lie algebra.

As is well known, the argument shift method is one of the effective approaches for constructing conserved quantities in integrable systems.

The Degree of Development of the Research Topic

The argument shift method for constructing Poisson commutative subalgebras in Poisson algebras was first proposed by Mishchenko and Fomenko [1] (generalizing the results of Manakov [2]). The applications of this method include constructing families of functions on the dual space g^* of a Lie algebra g, which are commutative with respect to the canonical Poisson structure on the symmetric algebra Sg of the Lie algebra g (commonly referred to as the Lie–Poisson structure). The primary data for this construction is a vector field ξ on the dual space g^* which is constant with respect to the standard affine coordinates on the dual space g^*. Later, Vinberg discovered that this construction could easily be transferred to an arbitrary Poisson manifold, where a "Nijenhuis" vector field is defined, and even to an arbitrary vector space with a bilinear operation and a "Nijenhuis" linear operator (see Section 1.2).

On the other hand, with a symplectic or Poisson manifold M, a noncommutative algebra, a "quantization" (geometric or deformation) of this manifold is associated. For example, in the case of deformation quantization, one can use the constructions of Fedosov [3] or Kontsevich [4]. As is well known, for any Lie algebra g, the quantization of the dual space g^* is closely related to the universal enveloping algebra Ug: one could say that if we restrict this construction to polynomial functions on the dual space g^*, i.e., on the symmetric algebra Sg, then the result of the quantization is the universal enveloping algebra Ug. In this regard, Vinberg [5] formulated the problem of constructing "argument shift" subalgebras (sometimes referred to as "Mishchenko–Fomenko algebras") in universal enveloping algebras.

This problem has been actively investigated by various authors; significant progress was made by Tarasov [7]: based on the given M. Nazarov and Olshansky [6] described functions on the dual space $gl(d,\mathbb{C})^*$, invariant under a certain class of subgroups of the Lie group $\mathrm{GL}(d,\mathbb{C})$. Tarasov showed that for vector fields of the form $\xi = \sum_i \xi^i \frac{\partial}{\partial x_{ii}}$ (where x_{ij} are the standard coordinates on the Lie algebra $gl(d,\mathbb{C})$, corresponding to the matrix elements), the lifts $\sigma(\xi^k(I_p))$ of the elements $\xi^k(I_p)$ (here and below I_p are the coefficients of the universal characteristic polynomial) in the universal enveloping algebra $Ugl(d,\mathbb{C})$ commute with each other.

Later, an alternative construction of the Mishchenko-Fomenko subalgebras in the universal enveloping algebra was proposed by Rybnikov [9]: in this work, the universal enveloping algebra $U_n\widehat{gl(d,\mathbb{C})}$ at the critical level n of the Kac–Moody algebra $\widehat{gl(d,\mathbb{C})}$ — a central extension of the infinite dimensional current Lie algebra — is considered. The algebra $U_n\widehat{gl(d,\mathbb{C})}$ has a large central subalgebra, the Feigin–Frenkel algebra; Rybnikov constructed a family of homomorphisms $f_\xi\colon U_n\widehat{g} \to Ug$ parameterized by the element $\xi \in g$ and showed that f_ξ maps the generators of the Feigin–Frenkel algebra to elements that "cover" the iterated derivatives in the direction ξ from the generators of the center of the symmetric algebra $Sgl(d,\mathbb{C})$. An important advantage of this construction is that it is almost unchanged when applied to any semi-simple Lie algebra g.

The construction obtained by this method is quite complicated and related to the properties of the infinite dimensional Lie algebras. Nevertheless, Rybnikov's approach remains the primary method for constructing the quantum argument shift algebra. Recently, the generators of the Feigin-Frenkel center have been extensively studied, and through the substantial efforts of Molev, Yakimova, and other mathematicians [13, 14], explicit formulas for the generators of the Mishchenko–Fomenko algebras were obtained not only for the the Lie algebras $gl(d,\mathbb{C})$ (or $sl(d,\mathbb{C})$) but also for the Lie algebras of other series of semi-simple Lie algebras (the series B, C, and D). It should be noted that the computational methods are often related to “Yangians”—particular infinite dimensional Hopf algebras derived from Lie algebras, in which large families of commutative subalgebras (Bethe algebras) are found.

The Purpose and Objectives of the Study

The aim of the dissertation is to study quantum derivations introduced by Gurevich, Pyatov, and Saponov, primarily in connection with the theory of quantum argument shift algebras in the universal enveloping algebras Ug. To achieve this goal, the following subjects are studied:

1. Let e be the generating matrix of the Lie algebra $gl(d, \mathbb{C})$. The center of the universal enveloping algebra $U gl(d, \mathbb{C})$ is generated by the elements $\operatorname{tr} e$, $\dots$, $\operatorname{tr} e^d$. Therefore, it is required to find the quantum derivations of the matrix elements $(e^n)^i_j$.

2. Study the properties of the quantum argument shift operator, defined by means of the quantum derivations.

3. Test the hypothesis that the quantum argument shifts of any two central elements of the universal enveloping algebra $U gl(d, \mathbb{C})$ commute.

4. Find iterated quantum argument shifts of any central element of the universal enveloping algebra $U gl(d, \mathbb{C})$.

5. Test the hypothesis that the quantum argument shifts of any orders of any two central elements of the universal enveloping algebra $U gl(d, \mathbb{C})$ commute (the quantum version of the Mishchenko and Fomenko theorem for $g = gl(d, \mathbb{C})$).

6. Test the hypothesis that the quantum argument shift algebras are generated

by iterated quantum argument shifts of central elements of the universal enveloping algebra $Ugl(d,\mathbb{C})$.

The Main Propositions to be Defended

In the dissertation research, the tasks listed in the previous section were studied. As a result of this study, the following main results, which are presented for defense, were obtained:

- A description is provided for the quantum derivation of matrix elements of powers of the generating matrix of the Lie algebra $gl(d,\mathbb{C})$.

- A quantum version of the Mishchenko and Fomenko theorem for the Lie algebra $g = gl(d,\mathbb{C})$ has been formulated and proven, both for first-order shifts and in the general case.

- A description is provided for the quantum argument shift algebras in the universal enveloping algebra $Ugl(d,\mathbb{C})$ in terms of quantum argument shift operators.

- A description is provided for first- and second-order quantum argument shifts for an arbitrary central element of the universal enveloping algebra $Ugl(d,\mathbb{C})$.

- A commutative family of elements in the universal enveloping algebra $Ugl(d,\mathbb{C})$, which had not been previously described in the literature, is explicitly presented.

The Scientific Novelty of the Research

All the results of the dissertation are new and have not been previously encountered in the literature known to the author. They consist of the following:

1. An explicit formula has been obtained for the quantum derivation of the matrix elements $(e^n)^i_j$ of the powers of the generating matrix in the universal enveloping algebra $Ugl(d, \mathbb{C})$.

2. The quantum argument shift of an arbitrary central element of the universal enveloping algebra $Ugl(d, \mathbb{C})$ has been explicitly described.

3. The generators of the subalgebra generated by the quantum argument shifts of the central elements of the universal enveloping algebra $Ugl(d, \mathbb{C})$ have been determined.

4. It has been shown that the quantum argument shifts of any two central elements of the universal enveloping algebra $Ugl(d, \mathbb{C})$ commute with each other.

5. The second-order quantum argument shift of an arbitrary central element of the universal enveloping algebra $Ugl(d, \mathbb{C})$ has been explicitly described.

6. The generators of the subalgebra generated by second-order quantum argument shifts have been determined.

7. The quantum version of the Mishchenko and Fomenko theorem for $g = gl(d, \mathbb{C})$ has been proven.

8. It has been proven that the quantum argument shift algebras are generated by iterated quantum argument shifts of central elements of the universal enveloping algebra $Ugl(d, \mathbb{C})$.

Research Methodology and Methods

The dissertation employs linear and general algebra, combinatorics, methods of differential geometry, the theory of Lie groups and Lie algebras and their universal enveloping algebras, methods of deformation quantization, and methods of computational symbolic calculations.

Theoretical and Practical Significance of the Study

The work is theoretical in nature. The results obtained can be applied in the theory of integrable systems, including both classical Hamiltonian systems on Lie algebras and in the theory of quantum integrable systems (in studying examples of such systems and establishing connections between them), as well as in the theory of Lie algebras and groups (for constructing commutative subalgebras in universal enveloping algebras). The theory represents an open problem, potentially linking the theory of Lie algebras, the theory of Yangians and quantum groups with the geometry of Lie groups and the theory of integrable systems. Additionally, the results can be used in problems of linear and general algebra and combinatorics.

Degree of Reliability

All results of the dissertation are original, substantiated by rigorous mathematical proofs, and have been published in open-access publications.

The results of other authors used in the dissertation are appropriately cited.

The Degree of Validity and Approbation of the Research Results

The results of the thesis were obtained while the author was studying at PhD program at the department of Differential Geometry and Applications at Moscow State University. The main results of this work were published in 3 papers [17, 18, 19] that appeared in the journals indexed by Web of Science, Scopus and RSCI. The auther also published 2 papers [20, 21] on the topic of the thesis.

The main results of the dissertation research have been reported and discussed at Russian and international scientific conferences:

- 3rd International Conference on Integrable Systems & Nonlinear Dynamics, Yaroslavl, Russia, October 7, 2021;
- 30th International Scientific Conference for Undergraduate and Graduate Students and Young Scientists "Lomonosov", Moscow, Russia, April 19, 2023;
- XL Workshop on Geometric Methods in Physics, Białowieża, Poland, July 6, 2023;
- XII International Symposium on Quantum Theory and Symmetries, Prague, Czech Republic, July 28, 2023;
- 4th International Conference on Integrable Systems & Nonlinear Dynamics, Yaroslavl, Russia, September 28, 2023;
- 20th Mathematics Conference for Young Researchers, Sapporo, Japan, March 4, 2024;
- Noncommutative Integrable Systems, Nagoya, Japan, March 13, 2024;

The results were also presented at well-known scientific seminars:

- "Modern geometric methods" under the supervision of Professor A.S. Mishchenko and Academician A.T. Fomenko at Moscow State University (repeatedly),
- "Noncommutative geometry and topology" under the supervision of Professor A.S. Mishchenko at Moscow State University (repeatedly),
- "Deformation quantization and quantum groups" under the supervision of Professor A.B. Zheglov and Associate Professor G.I. Sharygin at Independet University of Moscow (repeatedly).

The Structure of the Dissertation

This thesis consists of this introduction and four chapters. The total number of pages is 90, the list of references contains 21 items.

Chapter 1

In the first chapter, I will explain the origins of the theme, classical results, and the preliminaries required in the subsequent chapters. They include Poisson algebras and manifolds, Hamiltonian systems, the Lie–Poisson bracket on the algebra $C^\infty(g^*)$ (where g is a Lie algebra), the argument shift method (both the general form and for the Lie–Poisson bracket), universal enveloping algebras, the Poincaré-Birkhoff-Witt theorem, and its consequence.

Chapter 2

In the second chapter, we study the quantum derivation ∂^i_j of the universal enveloping algebra $U gl(d, \mathbb{C})$, introduced by Gurevich, Pyatov and Saponov. It is a quantum analogue of the set of partial derivatives on the symmetric algebra $Sgl(d,\mathbb{C}) = \mathbb{C}\big[e^i_j : i,j = 1,\ldots,d\big]$. It is characterized by the condition that it verifies the twisted version of the Leibniz rule (see formula (2.4)) and coincides with the usual partial derivatives on the set of standard generators of the Lie algebra $gl(d,\mathbb{C})$ (see formula (2.1)). Let us denote by ∂_ξ the following linear combination of the operators ∂^i_j:

$$\partial_\xi = \sum_{i,j=1}^{d} \xi^i_j \partial^j_i$$

Then the first result of this thesis is Theorem 2.4.1, which gives an explicit formula for the action of ∂_ξ on certain elements in the universal enveloping algebra $U gl(d,\mathbb{C})$, in particular on the central elements of the form $\operatorname{tr} e^n$, where e is the matrix of generators. It is used to prove the commutativity of the first degree quantum argument shifts of central elements in the universal enveloping algebra $U gl(d,\mathbb{C})$ and also the generators of the subalgebra generated by the quantum argument shifts up to the first order.

Chapter 3

In the third chapter, I prove the main conjecture, i.e. the commutativity of iterated quantum argument shifts of the central elements in the universal enveloping algebras $U gl(d,\mathbb{C})$. Here the key concept is provided by Vinberg [5] and Rybnikov [8]. Considering the adjoint action of the Lie group $GL(d,\mathbb{C})$ on the Lie algebra $gl(d,\mathbb{C})$, we may assume that the corresponding matrix

$$\begin{pmatrix} \xi(e^1_1) & \ldots & \xi(e^1_d) \\ \multicolumn{3}{c}{\ldots\ldots\ldots\ldots\ldots} \\ \xi(e^d_1) & \ldots & \xi(e^d_d) \end{pmatrix}$$

is diagonal and has d distinct eigenvalues. In this case, Vinberg and Rybnikov showed that the quantum argument shift algebra in the direction $\xi = \mathrm{diag}(z_1, \ldots, z_d)$ coincides with the commutant of the set

$$\left\{ e_i^i, \sum_{j \neq i} \frac{e_i^j e_j^i}{z_i - z_j} \right\}_{i=1}^{d}.$$

Now we are reduced to show

$$[e_i^i, \partial_\xi^n x] = [\sum_{j \neq i} \frac{e_i^j e_j^i}{z_i - z_j}, \partial_\xi^n x] = 0, \qquad i = 1, \ldots, d$$

for any central element x and positive integer n by induction. The base case ($n = 1$) is done by Theorem 2.4.1 and the inductive step is reduced to showing

$$[\mathrm{ad}\, e_i^i, \partial_\xi] = [[\mathrm{ad} \sum_{j \neq i} \frac{e_i^j e_j^i}{z_i - z_j}, \partial_\xi], \partial_\xi] = 0, \qquad 1, \ldots, d.$$

This is done by calculating concretely.

As a consequence of the main theorem, the algebra C_ξ generated by the set $\bigcup_{n=0}^{\infty} \partial_\xi^n C$ (where C is the center of the universal enveloping algebra $Ugl(d, \mathbb{C})$) turns out the quantum argument shift algebra in the direction ξ for any element ξ of the dual space $gl(d, \mathbb{C})^*$.

Chapter 4

And finally in the fourth chapter, I suggest a formula for the second degree of quantum argument shifts of the central elements in $Ugl(d, \mathbb{C})$; it gives a new previously unknown collection of commuting elements in this algebra. The precise formula for the second degree of quantum argument shifts of the central elements is

$$\partial_\xi^2 \prod_m \operatorname{tr} e^{n_m} = \sum_{m_1=-1}^{n_1} \operatorname{tr} e^{m_1} \sum_{m_2=-1}^{n_2} \operatorname{tr} e^{m_2} \cdots \sum_{k_1=-1}^{n_1-m_1-1} \sum_{k_2=-1}^{n_2-m_2-1} \cdots$$
$$\operatorname{tr}\Big(\xi \prod_\ell f_-^{(k_\ell)}(e) \partial \operatorname{tr}\big(\xi \prod_\ell f_-^{(n_\ell-m_\ell-k_\ell-2)}(e)\big)\Big),$$

$$\operatorname{tr}\Big(\xi e^m \partial \operatorname{tr}\big(\xi e^n\big)\Big) = \tau_\xi\Big(P_n^{(m)}\Big) + \sum_{k=0}^{n} \operatorname{tr}\big(\xi e^m f^{(n-k-1)}(e)\big) \operatorname{tr}\big(\xi e^k\big),$$

where ∂ is the modified quantum derivation (see Section 4.1) and we adopt the convention $\operatorname{tr} e^{-1} = 1$ and $f_-^{(-1)}(x) = 1$. This is also a consequence of Theorem 2.4.1. As a result, the algebra $C_\xi^{(2)}$ generated by the set $\bigcup_{n=0}^{2} \partial_\xi^n C$ is generated by the algebra $C_\xi^{(1)}$ and the set

$$\Big\{ \tau_\xi\Big(P_n^{(m)}\Big) + \tau_\xi\Big(P_m^{(n)}\Big) : m, n = 0, 1, 2, \ldots \Big\}. \tag{1}$$

I found the following linear relations

$$\tau_\xi\Big(P_{m+2n}^{(m)}\Big) + \tau_\xi\Big(P_m^{(m+2n)}\Big) = \sum_{k=0}^{n} \left(\binom{2n-k}{k} + \binom{2n-k-1}{k-1} \right) \tau_\xi\Big(P_{m+k}^{(m+k)}\Big)$$
$$\tau_\xi\Big(P_{m+2n+1}^{(m)}\Big) + \tau_\xi\Big(P_m^{(m+2n+1)}\Big) = \sum_{k=0}^{n} \binom{2n-k}{k} \Big(\tau_\xi\Big(P_{m+k+1}^{(m+k)}\Big) + \tau_\xi\Big(P_{m+k}^{(m+k+1)}\Big) \Big)$$

between the generators (1) using Mathematica. It turns out that the generators (1) are redundant and the algebra $C_\xi^{(2)}$ is generated by the algebra $C_\xi^{(1)}$ and the set

$$\Big\{ \tau_\xi\Big(P_n^{(n)}\Big), \tau_\xi\Big(P_{n+1}^{(n)}\Big) + \tau_\xi\Big(P_n^{(n+1)}\Big) : n = 1, 2, \ldots \Big\}.$$

Chapter 1

Preliminary

1.1 Poisson Structures and Hamiltonian Systems

1.1.1 Hamilton Systems on Euclidean Spaces

Without much doubt, one can say that the classical theory of Hamiltonian systems begins with the following definition:

Definition 1.1.1. For any pair of smooth functions f, g on the phase space $\mathbb{R}^n \times \mathbb{R}^n$ (where the first copy of the Euclidean space corresponds to the coordinates $q^1, \dots, q^n$ and the second one represents the momenta $p_1, \dots, p_n$ of a system), their **Poisson bracket** is given by the formula

$$\{f, g\} = \sum_{k=1}^{n} \left(\frac{\partial f}{\partial p_k} \frac{\partial g}{\partial q^k} - \frac{\partial g}{\partial p_k} \frac{\partial f}{\partial q^k} \right).$$

Given such structure one can associate a Hamiltonian system with any smooth function H on the phase space: we put

$$\begin{cases} \dot{p}_i = -\frac{\partial H}{\partial q^i}, \\ \dot{q}_i = \frac{\partial H}{\partial p_i}. \end{cases}$$

In this case H is called the *Hamilton function* or simply *Hamiltonian* of the system. Also recall that two functions f, g are said to be *in involution*, if $\{f, g\} = 0$.

The crucial role in this theory is played by the Liouville integrability criterion:

Theorem 1.1.1 (Liouville's theorem)**.** *If we have a Hamilton system on the space $\mathbb{R}^{2n}$ with Hamiltonian H, then the integrability of the system (the existence of formulas that represent the solutions of the system in the form of iterated integrals) follows from the existence of n functionally independent first integrals $I_1 = H, I_2, \ldots, I_n$, involutive with respect to the Poisson bracket.*

One may say that the study of Hamilton systems on $\mathbb{R}^{2n}$ amounts to the study of commutative families of functions in the Poisson Lie algebra of functions on $\mathbb{R}^{2n}$, so that by a slight abuse of terminology one often calls such families (and the corresponding Poisson-commutative subalgebras in $C^\infty(\mathbb{R}^{2n})$) *integrable systems*.

1.1.2 Poisson Structures on Manifolds

The definitions from the previous section can be transferred directly from the case of the Euclidean space $\mathbb{R}^{2n}$ to any manifold, equipped with a bracket operation on its smooth functions:

$$\{,\} : C^\infty(M) \otimes C^\infty(M) \to C^\infty(M).$$

This operation should verify the conditions of skew-symmetry, Leibniz rule with respect to both arguments and Jacobi identity:

$$\begin{aligned}
\{f, g\} &= -\{g, f\} \\
\{f, gh\} &= \{f, g\}h + g\{f, h\} \\
\{f, \{g, h\}\} &= \{\{f, g\}, h\} + \{g, \{f, h\}\}.
\end{aligned}$$

The formula from definition 1.1.1 clearly verifies all this conditions;the corresponding structure is usually referred to as the *natural Poisson bracket, associated with the canonical symplectic structure on* $\mathbb{R}^{2n}$.

Manifolds, equipped with such brackets are called *Poisson manifolds.* It is easy to see (in local coordinates) that any skew symmetric bracket on a manifold that verifies the Leibniz rule, is determined by a formula

$$\{f, g\}(x) = \pi^{ij}(x)\frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j},$$

where $\pi^{ij}(x)$ are components of an anti-symmetric covariant tensor π on the manifold. Such tensors are called *bivectors*, they are sections of the exterior square of the tangent bundle of M. Jacobi identity implies an additional condition on π; this condition can be expressed as the terms of the so-called *Schouten-Nijenhuis bracket* on the space of skew-symmetric covariant tensors, that extends the usual commutator of vector fields. If $[,]$ denotes such operation, then a bivector field π (a smooth section of $\Lambda^2 T^M$) will induce a bracket verifying the Jacobi identity iff the following condition holds

$$[\pi, \pi] = 0,$$

In this case we will say that (M, π) is a *Poisson manifold.* We will call any subalgebra A in $C^\infty(M)$, commutative with respect to $\{,\}$ *(pre)integrable system on* M. Observe that that we omit the usual condition that the algebra A is large: the proper concept of integrability should contain this condition, therefore we will sometimes use the prefix "pre".

1.1.2.1 Example: Canonical Structure on Coadjoint Representation Space

The main example of Poisson structure for us will be the following construction, called the Lie–Poisson structure. Let g be a Lie algebra, and g^* its dual space.

Let f, g be two smooth functions on g^*. Then

$$\{f, g\} = [e_i, e_j] \frac{\partial f}{\partial e_i} \frac{\partial g}{\partial e_j}$$

defines a unique Poisson structure on $C^\infty(g^*)$, where $\{e_1, \dots, e_n\}$ is a linear basis of g.

1.2 Argument Shift Method

Various constructions of integrable systems of this sort, as well as the study of their geometric, topological etc. properties make up an important part of modern Mathematics, including the theory of Differential equations, Geometry and Algebra. On the other hand since the dawn of the XXth century, with the development of Quantum Physic, this theory has evolved to embrace its noncommutative analog, called the theory of *Quantum integrable systems.*

There exist several standard constructions of integrable systems, and one of the most intriguing questions in the quantum theory is whether some of these methods can be used (after appropriate modifications) to obtain quantum integrable systems. One of these classical methods is the argument shift method, which we study in the thesis.

1.2.1 General Construction

Let us briefly describe the method in full generality. Suppose that A is a vector space over a field of characteristic 0 and let ξ be a linear mapping and η be a bilinear mapping on the vector space A. We obtain another bilinear mapping $[\xi, \eta]$ on the vector space A by the equation

$$[\xi, \eta](x, y) = \xi\eta(x, y) - \eta(\xi x, y) - \eta(x, \xi y).$$

Suppose that we have

$$[\xi, [\xi, \eta]] = 0$$

and let x and y be elements of the vector space A such that we have

$$\eta(x, z) = \eta(z, y) = 0$$

for any element z of the vector space A.

1. We have

$$\begin{aligned}
0 &= \xi\eta(x, y) \\
&= [\xi, \eta](x, y) + \eta(\xi x, y) + \eta(x, \xi y) \\
&= [\xi, \eta](x, y), \\
0 &= \xi[\xi, \eta](x, y) \\
&= [\xi, [\xi, \eta]](x, y) + [\xi, \eta](\xi x, y) + [\xi, \eta](x, \xi y) \\
&= [\xi, \eta](\xi x, y) + [\xi, \eta](x, \xi y), \\
0 &= \xi\eta(\xi x, y) \\
&= [\xi, \eta](\xi x, y) + \eta(\xi^2 x, y) + \eta(\xi x, \xi y) \\
&= [\xi, \eta](\xi x, y) + \eta(\xi x, \xi y), \\
0 &= \xi\eta(x, \xi y) \\
&= [\xi, \eta](x, \xi y) + \eta(\xi x, \xi y) + \eta(x, \xi^2 y) \\
&= [\xi, \eta](x, \xi y) + \eta(\xi x, \xi y)
\end{aligned}$$

since we have

$$\eta(x, y) = \eta(\xi x, y) = \eta(x, \xi y) = \eta(\xi^2 x, y) = \eta(x, \xi^2 y) = 0.$$

We have

$$\begin{aligned} 0 &= [\xi,\eta](\xi x, y) + [\xi,\eta](x,\xi y) \\ &= [\xi,\eta](\xi x, y) + \eta(\xi x, \xi y) \\ &= [\xi,\eta](x, \xi y) + \eta(\xi x, \xi y) \end{aligned}$$

and $\eta(\xi x, \xi y) = 0$ since the underlying field is of characteristic 0.

2. We have $[\xi,\eta](\xi x, y) = [\xi,\eta](x,\xi y) = 0$ and

$$\begin{aligned} 0 &= \xi[\xi,\eta](\xi x, y) \\ &= [\xi,[\xi,\eta]](\xi x, y) + [\xi,\eta](\xi^2 x, y) + [\xi,\eta](\xi x, \xi y) \\ &= [\xi,\eta](\xi^2 x, y) + [\xi,\eta](\xi x, \xi y), \\ 0 &= \xi[\xi,\eta](x, \xi y) \\ &= [\xi,[\xi,\eta]](x, \xi y) + [\xi,\eta](\xi x, \xi y) + [\xi,\eta](x, \xi^2 y) \\ &= [\xi,\eta](\xi x, \xi y) + [\xi,\eta](x, \xi^2 y), \\ 0 &= \xi\eta(\xi^2 x, y) \\ &= [\xi,\eta](\xi^2 x, y) + \eta(\xi^3 x, y) + \eta(\xi^2 x, \xi y) \\ &= [\xi,\eta](\xi^2 x, y) + \eta(\xi^2 x, \xi y), \\ 0 &= \xi\eta(\xi x, \xi y) \\ &= [\xi,\eta](\xi x, \xi y) + \eta(\xi^2 x, \xi y) + \eta(\xi x, \xi^2 y), \\ 0 &= \xi\eta(x, \xi^2 y) \\ &= [\xi,\eta](x, \xi^2 y) + \eta(\xi x, \xi^2 y) + \eta(x, \xi^3 y) \\ &= [\xi,\eta](x, \xi^2 y) + \eta(\xi x, \xi^2 y). \end{aligned}$$

We have

$$\begin{aligned}0 &= [\xi,\eta](\xi^2x,y) + [\xi,\eta](\xi x,\xi y)\\ &= [\xi,\eta](\xi x,\xi y) + [\xi,\eta](x,\xi^2y)\\ &= [\xi,\eta](\xi^2x,y) + \eta(\xi^2x,\xi y)\\ &= [\xi,\eta](\xi x,\xi y) + \eta(\xi^2x,\xi y) + \eta(\xi x,\xi^2y)\\ &= [\xi,\eta](x,\xi^2y) + \eta(\xi x,\xi^2y)\end{aligned}$$

and

$$\eta(\xi^2x,\xi y) = -\eta(\xi^2x,\xi y) - \eta(\xi x,\xi^2y) = \eta(\xi x,\xi^2y).$$

We have

$$\eta(\xi^2x,\xi y) = \eta(\xi x,\xi^2y) = 0$$

since the underlying field is of characteristic 0.

The following generalization of the Mischenko and Fomenko result is often attributed to Vinberg, but I could not find a good reference for it.

Theorem 1.2.1. *Suppose that ξ is a linear mapping and let η be a bilinear mapping on a vector space over a field of characteristic 0 such that we have*

$$[\xi,[\xi,\eta]] = 0.$$

Suppose that x and y are elements of the vector space such that we have

$$\eta(x,z) = \eta(z,y) = 0$$

for any element z of the vector space. We have $\eta(\xi^mx,\xi^ny) = 0$ for any positive integers m and n.

Proof. The proof is by induction on the positive integer $m+n$ and we have

$$\eta(\xi^{m-1}x,\xi^{n-1}y) = \eta(\xi^mx,\xi^{n-1}y) = \eta(\xi^{m-1}x,\xi^ny) = 0$$

by the induction hypothesis. We have

$$
\begin{aligned}
0 &= \xi^2\eta(\xi^{m-1}x, \xi^{n-1}y) \\
&= \xi[\xi,\eta](\xi^{m-1}x, \xi^{n-1}y) \\
&= [\xi,\eta](\xi^{m}x, \xi^{n-1}y) + [\xi,\eta](\xi^{m-1}x, \xi^{n}y), \\
0 &= \xi\eta(\xi^{m}x, \xi^{n-1}y) + \xi\eta(\xi^{m-1}x, \xi^{n}y) \\
&= \eta(\xi^{m+1}x, \xi^{n-1}y) + 2\eta(\xi^{m}x, \xi^{n}y) + \eta(\xi^{m-1}x, \xi^{n+1}y) \qquad (1.1)
\end{aligned}
$$

and

$$
\begin{aligned}
&\eta(x, \xi^{m+n}y) = 0, \\
&\eta(\xi x, \xi^{m+n-1}y), \\
&\eta(\xi^2 x, \eta^{m+n-2}y) = -2\eta(\xi x, \xi^{m+n-1}y), \\
&\eta(\xi^3 x, \eta^{m+n-3}y) = -2\eta(\xi^2 x, \xi^{m+n-2}y) - \eta(\xi x, \xi^{m+n-1}y) = 3\eta(\xi x, \xi^{m+n-1}y), \ldots, \\
&\eta(\xi^m x, \xi^n y) = -(-1)^m m\eta(\xi x, \xi^{m+n-1}y), \ldots, \\
&\eta(\xi^{m+n}x, y) = -(-1)^{m+n}(m+n)\eta(\xi x, \xi^{m+n-1}y) = 0
\end{aligned}
$$

by the recursion formula (1.1). We have

$$\eta(\xi x, \xi^{m+n-1}y) = 0$$

since the underlying field is of characteristic 0. □

1.2.2 Argument Shift Method on the Dual Space of a Lie Algebra

We apply it to the dual space of a Lie algebra. Suppose that g is a finite dimensional Lie algebra over a field of characteristic 0 and we write η for the Lie–Poisson bracket on the symmetric algebra Sg. Suppose that ξ is an element of the dual space g^*. We identify the element ξ with the unique derivation on the symmetric algebra Sg extending the linear functional ξ. We have

$$\begin{aligned}\xi &= \xi(e_1)e^1 + \cdots + \xi(e_d)e^d \\ &= \xi(e_1)\frac{\partial}{\partial e_1} + \cdots + \xi(e_d)\frac{\partial}{\partial e_d}.\end{aligned}$$

Theorem 1.2.2 (Mishchenko and Fomenko [1])**.** *Suppose that x and y are Poisson central elements of the symmetric algebra Sg. We have $\eta(\xi^m x, \xi^n y) = 0$ for any positive integers m and n.*

Proof. We have

$$\begin{aligned}[\xi,\eta] &= \sum_{i,j,k=1}^{d} \xi(e_k)[\frac{\partial}{\partial e_k}, [e_i,e_j]\frac{\partial}{\partial e_i} \otimes \frac{\partial}{\partial e_j}] \\ &= \sum_{i,j,k=1}^{d} \xi(e_k)\frac{\partial [e_i,e_j]}{\partial e_k}\frac{\partial}{\partial e_i} \otimes \frac{\partial}{\partial e_j} = \sum_{i,j=1}^{d} \xi[e_i,e_j]\frac{\partial}{\partial e_i} \otimes \frac{\partial}{\partial e_j}\end{aligned}$$

and

$$[\xi,[\xi,\eta]] = \sum_{i,j,k=1}^{d} \xi(e_k)\xi[e_i,e_j][\frac{\partial}{\partial e_k}, \frac{\partial}{\partial e_i} \otimes \frac{\partial}{\partial e_j}] = 0. \quad \square$$

Definition 1.2.1. We write $\bar{C}$ for the Poisson center of the symmetric algebra Sg. The Poisson commutative subalgebra $\bar{C}_\xi$ generated by the subset $\bigcup_{n=0}^{\infty} \xi^n \bar{C}$ is called the argument shift algebra in the direction ξ.

1.2.3 Case $g = gl(d,\mathbb{C})$

Suppose that $\left\{ e^i_j : i,j = 1,\dots,d \right\}$ is a linear basis of the general linear Lie algebra $gl(d,\mathbb{C})$ satisfying the commutation relation $\left[e^{i_1}_{j_1}, e^{i_2}_{j_2}\right] = e^{i_2}_{j_1}\delta^{i_1}_{j_2} - \delta^{i_2}_{j_1}e^{i_1}_{j_2}$. The Poisson center $\bar{C}$ of the symmetric algebra $Sgl(d,\mathbb{C})$ is freely generated by the finite sequence $\left(\operatorname{tr} e, \dots, \operatorname{tr} e^d\right)$. The argument shift algebra $\bar{C}_\xi$ in the direction

ξ is generated by the finite set $\{\, \xi^m \operatorname{tr} e^n : 0 \le m < n \le d \,\}$ by the Leibniz rule and we have

$$\xi^m \operatorname{tr} e^n = \frac{1}{(n-m)!} \sum_{\sigma \in S_n} \operatorname{tr}\bigl(\zeta_{\sigma(1)} \cdots \zeta_{\sigma(n)}\bigr), \qquad \zeta_k = \begin{cases} \xi & k \le m \\ e & k > m \end{cases}.$$

1.3 Universal Enveloping Algebras and Vinberg's Problem

A deformation quantisation of a Poisson manifold M is a sequence of bidifferential operators $B_1(x,y)$, $B_2(x,y)$, ... such that the star product

$$x \star y = xy + (i\hbar) B_1(x,y) + (i\hbar)^2 B_2(x,y) + \cdots$$

is associative and satisfies the condition

$$B_1(x,y) - B_1(y,x) = \left[\frac{x \star y - y \star x}{i\hbar} \right]_{i\hbar = 0} = \{x, y\}.$$

This condition is inspired by the observation of Dirac

$$\lim_{i\hbar \to 0} \frac{xy - yx}{i\hbar} = \{x, y\},$$

where x and y of the left-hand side are the quantum analogues of the classical observable functions x and y of the right-hand side. Kontsevich [4] finally settled the deformation quantisation problem and the degrees of his star product for the Lie–Poisson bracket of a Lie algebra g are finite on the symmetric algebra $S(g)$. We can define the star product on the symmetric algebra $S(g)$ putting $i\hbar = 1$. The universal enveloping algebra $U(g)$ is isomorphic to the symmetric algebra $S(g)$ equipped with the star product and can be regarded as a deformation quantisation.

1.3.1 Definition and the Universal Property

Suppose that $T(g)$ is the tensor algebra of a Lie algebra g.

Definition 1.3.1. We define the universal enveloping algebra

$$U(g) = \frac{T(g)}{\left(\, xy - yx - [x,y] : x, y \in g \,\right)},$$

where the denominator denotes the ideal generated by the set

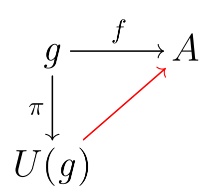
$$\left\{\, xy - yx - [x,y] : x, y \in g \,\right\}.$$

We write π for the canonical epimorphism of the tensor algebra $T(g)$ onto the universal enveloping algebra $U(g)$.

Proposition 1.3.1. *The restriction of the mapping π to the Lie algebra g is a homomorphism of Lie algebras.*

The universal enveloping algebra is characterised by the following universal property.

Theorem 1.3.1. *Suppose that f is a homomorphism of Lie algebras of the Lie algebra g into a unital algebra A. There exists a unique homomorphism of unital algebras of the universal enveloping algebra $U(g)$ into the unital algebra A such that the following diagram commutes.*



1.3.2 Poincaré-Birkhoff-Witt Theorem and the Consequence

Theorem 1.3.2 (Poincaré-Birkhoff-Witt)**.** *We have the following.*

1. *The canonical homomorphism of the Lie algebra g into the universal enveloping algebra $U(g)$ is an imbedding.*

2. *The linear mapping f of the symmetric algebra $S(g)$ onto the universal enveloping algebra $U(g)$ defined by*

$$f(x_1 \cdots x_n) = x_1 \cdots x_n$$

for any nonnegative integer n and for any element x of the set

$$\{\, x \in X^n : x_1 \leq \cdots \leq x_n \,\}$$

is a linear isomorphism.

Definition 1.3.2. We define $U_n(g) = \{0\}$ for any negative integer n and we define

$$\operatorname{gr}_n U(g) = \frac{U_n(g)}{U_{n-1}(g)}$$

for any integer n.

The following is a consequence of the Poincaré-Birkhoff-Witt theorem.

Theorem 1.3.3. *The mapping*

$$\begin{aligned} &\operatorname{gr}_m U(g) \times \operatorname{gr}_n U(g) \to \operatorname{gr}_{m+n} U(g), \\ &\qquad \big(x + U_{m-1}(g), y + U_{n-1}(g)\big) \mapsto xy + U_{m+n-1}(g) \end{aligned} \tag{1.2}$$

is bilinear for any nonnegative integers m and n and the direct sum

$$\operatorname{gr} U(g) = \bigoplus_{n=0}^{\infty} \operatorname{gr}_n U(g)$$

is a graded commutative algebra with identity isomorphic to the symmetric algebra $S(g)$.

Proof. We show that the graded algebra $\operatorname{gr} U(g)$ is commutative. It is sufficient to show that the multiplication (1.2) is commutative. We may assume $(x, y) = (x_1 \cdots x_m, y_1 \cdots y_n)$ for some element $(x_1, \ldots, x_m, y_1 \ldots, y_n)$ of the product g^{m+n} since the mapping (1.2) is bilinear. We have

$$\begin{aligned}
\big(x + U_{m-1}(g)\big)\big(y + U_{n-1}(g)\big) &= x_1 \cdots x_m y_1 \cdots y_n + U_{m+n-1}(g) \\
&= x_1 \cdots x_{m-1} y_1 x_m y_2 \cdots y_n \\
&\qquad + x_1 \cdots x_{m-1} [x_m, y_1] y_2 \cdots y_n + U_{m+n-1}(g) \\
&= x_1 \cdots x_{m-1} y_1 x_m y_2 \cdots y_n + U_{m+n-1}(g) = \cdots \\
&= y_1 \cdots y_n x_1 \cdots x_m + U_{m+n-1}(g) \\
&= \big(y + U_{n-1}(g)\big)\big(x + U_{m-1}(g)\big).
\end{aligned}$$

The inclusion mapping of the Lie algebra $g = \operatorname{gr}_1 U(g)$ into the unital commutative algebra $\operatorname{gr} U(g)$ extends uniquely to the unital algebraic homomorphism of the symmetric algebra $S(g)$ into the unital commutative algebra $\operatorname{gr} U(g)$ by the universal property of the symmetric algebra $S(g)$.

$$\begin{array}{ccc} g & \longrightarrow & \operatorname{gr} U(g) \\ \downarrow & \nearrow & \\ S(g) & & \end{array} \tag{1.3}$$

The homomorphism (1.3) is graded since we have

$$\begin{aligned}
S^n(g) \to \operatorname{gr} U(g), \qquad x_1 \cdots x_n \mapsto &\big(x_1 + U_0(g)\big) \cdots \big(x_n + U_0(g)\big) \\
&= x_1 \cdots x_n + U_{n-1}(g) \in \operatorname{gr}_n U(g)
\end{aligned} \tag{1.4}$$

for any element $(x_1, \ldots, x_n)$ of the product g^n. The epimorphism (1.3) is an isomorphism since the epimorphism (1.4) of the vector space $S^n(g)$ onto the vector space $\operatorname{gr}_n U(g)$ is a monomorphism by the Poincaré-Birkhoff-Witt theorem. □

The mapping

$$\begin{aligned}\operatorname{gr}_m U(g) \times \operatorname{gr}_n U(g) &\to \operatorname{gr}_{m+n-1} U(g),\\ \big(x + U_{m-1}(g), y + U_{n-1}(g)\big) &\mapsto \big\{x + U_{m-1}(g), y + U_{n-1}(g)\big\}\\ &= [x, y] + U_{m+n-2}(g) \qquad (1.5)\end{aligned}$$

is well-defined and bilinear since the set $\big[U_m(g), U_n(g)\big]$ is contained in the subspace $U_{m+n-1}(g)$ for any nonnegative integers m and n. The bilinear mappings (1.5) define a Poisson bracket on the graded algebra $\operatorname{gr} U(g)$ and it coincides with the Lie–Poisson bracket on the symmetric algebra $S(g) = \operatorname{gr} U(g)$ since we have

$$\big\{x + U_0(g), y + U_0(g)\big\} = [x, y] + U_0(g)$$

for any elements x and y of the Lie algebra g.

Chapter 2

Quantum Derivation of $Ugl(d,\mathbb{C})$

2.1 Introduction

The purpose of this dissertation is to describe the quantum argument shift algebra for an element ξ of the dual space $gl(d,\mathbb{C})^*$ of the Lie algebra $gl(d,\mathbb{C})$ in terms of a quantum argument shift operator ∂_ξ and to clarify the properties of this operator. The quantum derivation ∂, introduced by Gurevich, Pyatov, and Saponov, plays an essential role in this context.

We begin with notation. Suppose that d is a nonnegative integer. We write x^i_j for the (i,j) elements of a d by d matrix x and let

$$x^i = \begin{pmatrix} x^i_1 & \cdots & x^i_d \end{pmatrix}, \qquad\qquad x_j = \begin{pmatrix} x^1_j \\ \vdots \\ x^d_j \end{pmatrix}$$

denote the row and the column vectors. We write δ for the d by d identity matrix: the symbols δ^i and δ_j denote the unit i-th row vector and the unit j-th column vector respectively. We write

$$E^i_j = \delta_j \delta^i = \begin{pmatrix} \delta^1_j \delta^i_1 & \dots & \delta^1_j \delta^i_d \\ \dots & \dots & \dots \\ \delta^d_j \delta^i_1 & \dots & \delta^d_j \delta^i_d \end{pmatrix}$$

for the (j,i) matrix unit. We have the commutation relation

$$\left[E^{i_1}_{j_1}, E^{i_2}_{j_2} \right] = E^{i_2}_{j_1} \delta^{i_1}_{j_2} - \delta^{i_2}_{j_1} E^{i_1}_{j_2}.$$

We consider the general linear Lie algebra $gl(d, \mathbb{C})$. We write

- $T gl(d, \mathbb{C})$ for the tensor algebra
- $U gl(d, \mathbb{C})$ for the universal enveloping algebra
- $S gl(d, \mathbb{C})$ for the symmetric algebra

of the Lie algebra $gl(d, \mathbb{C})$. We have

$$T gl(d, \mathbb{C}) = \bigoplus_{n=0}^{\infty} gl(d, \mathbb{C})^{\otimes n}, \qquad S gl(d, \mathbb{C}) = \bigoplus_{n=0}^{\infty} S^n gl(d, \mathbb{C}),$$

where the symbol $S^n gl(d, \mathbb{C})$ denote the n-th symmetric power of the Lie algebra $gl(d, \mathbb{C})$.

Definition 2.1.1. Suppose that S is a subset of a unital complex algebra. We write (S) for the ideal generated by the subset S.

We have

$$U gl(d, \mathbb{C}) = \frac{T gl(d, \mathbb{C})}{\left(\left\{ \, xy - yx - [x, y] : x, y \in gl(d, \mathbb{C}) \, \right\} \right)}$$

and

$$S gl(d, \mathbb{C}) = \frac{T gl(d, \mathbb{C})}{\left(\left\{ \, xy - yx : x, y \in gl(d, \mathbb{C}) \, \right\} \right)}.$$

We write e for the d by d matrix composed of the indeterminates e^i_j and let

- $\mathbb{C}\langle e^i_j : i,j = 1,\dots,d\rangle$ be the free unital algebra
- $\mathbb{C}[e^i_j : i,j = 1,\dots,d]$ be the free unital commutative algebra

on the indeterminates e^i_j. We have

- $Tgl(d,\mathbb{C}) = \mathbb{C}\langle e^i_j : i,j = 1,\dots,d\rangle$.
- $Ugl(d,\mathbb{C}) = \dfrac{\mathbb{C}\langle e^i_j : i,j = 1,\dots,d\rangle}{\Big(\Big\{ e^{i_1}_{j_1} e^{i_2}_{j_2} - e^{i_2}_{j_2} e^{i_1}_{j_1} - \delta^{i_1}_{j_2} e^{i_2}_{j_1} + e^{i_1}_{j_2} \delta^{i_2}_{j_1} : i_1, j_1, i_2, j_2 = 1,\dots,d \Big\}\Big)}$.
- $$\begin{aligned} Sgl(d,\mathbb{C}) &= \frac{\mathbb{C}\langle e^i_j : i,j = 1,\dots,d\rangle}{\Big\{ e^{i_1}_{j_1} e^{i_2}_{j_2} - e^{i_2}_{j_2} e^{i_1}_{j_1} : i_1, j_1, i_2, j_2 = 1,\dots,d \Big\}} \\ &= \mathbb{C}[e^i_j : i,j = 1,\dots,d]. \end{aligned}$$

The set $\{ e^i_j : i,j = 1,\dots,d \}$ is a basis of the Lie algebra $gl(d,\mathbb{C})$ and we have the commutation relation

$$[e^{i_1}_{j_1}, e^{i_2}_{j_2}] = e^{i_2}_{j_1} \delta^{i_1}_{j_2} - \delta^{i_2}_{j_1} e^{i_1}_{j_2}. \tag{2.1}$$

Definition 2.1.2. Suppose that S is a set. We write $M(d,S)$ for the set of d by d matrices with entries in the set S.

We call the matrix e the generating matrix of the Lie algebra $gl(d,\mathbb{C})$ and regard it as the element of the set $M\big(d, gl(d,\mathbb{C})\big)$. We define the derivations

$$\bar{\partial}^j_i = \frac{\partial}{\partial e^i_j}, \qquad i,j = 1,\dots,d$$

on the symmetric algebra $Sgl(d,\mathbb{C}) = \mathbb{C}[e^i_j : i,j = 1,\dots,d]$ and let

$$\bar{\partial} = \begin{pmatrix} \bar{\partial}^1_1 & \dots & \bar{\partial}^1_d \\ \dots & \dots & \dots \\ \bar{\partial}^d_1 & \dots & \bar{\partial}^d_d \end{pmatrix} \in \hom\Big(Sgl(d,\mathbb{C}), M\big(d, Sgl(d,\mathbb{C})\big)\Big).$$

We have

$$\bar\partial x = \begin{pmatrix} \bar\partial^1_1 x & \dots & \bar\partial^1_d x \\ \multicolumn{3}{c}{\dotfill} \\ \bar\partial^d_1 x & \dots & \bar\partial^d_d x \end{pmatrix}$$

for any element x of the symmetric algebra $Sgl(d,\mathbb{C})$. The mapping $\bar\partial$ is the unique linear mapping of the symmetric algebra $Sgl(d,\mathbb{C})$ into the matrix algebra $M\big(d, Sgl(d,\mathbb{C})\big)$ satisfying the following.

1. $\bar\partial\nu = 0$ for any scalar ν.

2. $\bar\partial\,\mathrm{tr}(\xi e) = \xi$ for any numerical matrix ξ.

3. (Leibniz rule)

$$\bar\partial(xy) = (\bar\partial x)y + x(\bar\partial y) \tag{2.2}$$

for any elements x and y of the symmetric algebra $Sgl(d,\mathbb{C})$.

We have the identification

$$gl(d,\mathbb{C})^* = M(d,\mathbb{C}), \qquad\qquad \xi = \begin{pmatrix} \xi(e^1_1) & \dots & \xi(e^1_d) \\ \multicolumn{3}{c}{\dotfill} \\ \xi(e^d_1) & \dots & \xi(e^d_d) \end{pmatrix}.$$

Suppose that ξ is a linear functional on the Lie algebra $gl(d,\mathbb{C})$. The linear mapping

$$\bar\partial_\xi = \mathrm{tr}(\xi\bar\partial) = \sum_{i,j=1}^{d} \xi(e^i_j)\frac{\partial}{\partial e^i_j}$$

is a unique derivation on the symmetric algebra $Sgl(d,\mathbb{C})$ extending the linear functional ξ:

$$\begin{array}{ccc} Sgl(d,\mathbb{C}) & \xrightarrow{\bar\partial_\xi} & Sgl(d,\mathbb{C}) \\ \uparrow & & \uparrow \\ gl(d,\mathbb{C}) & \xrightarrow{\xi} & \mathbb{C} \end{array}.$$

We have

$$\{\bar{\partial}_\xi^m x, \bar{\partial}_\xi^n y\} = 0$$

for any Poisson central elements x and y of the symmetric algebra $Sgl(d,\mathbb{C})$ and positive integers m and n by Theorem 1.2.2.

We would like to quantize the mapping $\bar{\partial}$. There is no such linear mapping ∂ of the universal enveloping algebra $Ugl(d,\mathbb{C})$ into the matrix algebra $M\big(d, Ugl(d,\mathbb{C})\big)$ since we have the contradiction

$$\begin{aligned} 0 &= \partial\Big(e^{i_1}_{j_1} e^{i_2}_{j_2} - e^{i_2}_{j_2} e^{i_1}_{j_1}\Big) && \text{(by the Leibniz rule)} \\ &= \partial\Big(\delta^{i_1}_{j_2} e^{i_2}_{j_1} - e^{i_1}_{j_2} \delta^{i_2}_{j_1}\Big) && \text{(by the commutation relation)} \\ &= \delta^{i_1}_{j_2} E^{i_2}_{j_1} - E^{i_1}_{j_2} \delta^{i_2}_{j_1} \neq 0 && \text{(by the condition 2)} \end{aligned} \tag{2.3}$$

if there is such a linear mapping ∂. We would consider the following condition instead of the Leibniz rule.

3. (quantum Leibniz rule)

$$\partial(xy) = (\partial x)y + x(\partial y) + (\partial x)(\partial y) \tag{2.4}$$

for any elements x and y of the universal enveloping algebra $Ugl(d,\mathbb{C})$.

We have

$$\begin{aligned} \big[E^{i_1}_{j_1}, E^{i_2}_{j_2}\big] &= E^{i_1}_{j_1} E^{i_2}_{j_2} - E^{i_2}_{j_2} E^{i_1}_{j_1} \\ &= (\partial e^{i_1}_{j_1})(\partial e^{i_2}_{j_2}) - (\partial e^{i_2}_{j_2})(\partial e^{i_1}_{j_1}) && \text{(by the condition 2)} \\ &= \partial\Big(e^{i_1}_{j_1} e^{i_2}_{j_2} - e^{i_2}_{j_2} e^{i_1}_{j_1}\Big) && \text{(by the quantum Leibniz rule)} \\ &= \partial\Big(\delta^{i_1}_{j_2} e^{i_2}_{j_1} - e^{i_1}_{j_2} \delta^{i_2}_{j_1}\Big) && \text{(by the commutation relation)} \\ &= \delta^{i_1}_{j_2} E^{i_2}_{j_1} - E^{i_1}_{j_2} \delta^{i_2}_{j_1}. && \text{(by the condition 2)} \end{aligned}$$

Gurevich, Pyatov, and Saponov proposed the quantum Leibniz rule and showed that there is a unique quantum derivation ∂.

2.2 Construction of the Quantum Derivation

We construct the quantum derivation ∂ in this section. We first construct it on the tensor algebra $Tgl(d,\mathbb{C})$.

Definition 2.2.1. We define a linear mapping

$$Tgl(d,\mathbb{C}) \to M\big(d, Tgl(d,\mathbb{C})\big), \qquad x \mapsto \partial x = (\partial^i_j x)^d_{i,j=1} \tag{2.5}$$

inductively as follows:

1. We define $\partial\nu = 0$ for any scalar ν.

2. We define

$$\begin{aligned}\partial^i_j\Big(e^{i_1}_{j_1}\cdots e^{i_n}_{j_n}\Big) = \partial^i_j\Big(e^{i_1}_{j_1}\cdots e^{i_{n-1}}_{j_{n-1}}\Big)e^{i_n}_{j_n} \\ + \Big(e^{i_1}_{j_1}\cdots e^{i_{n-1}}_{j_{n-1}}\delta^i_{j_n} + \partial^i_{j_n}\Big(e^{i_1}_{j_1}\cdots e^{i_{n-1}}_{j_{n-1}}\Big)\Big)\delta^{i_n}_j\end{aligned} \tag{2.6}$$

   for $n > 0$.

Proposition 2.2.1. *The linear mapping (2.5) satisfies the following.*

1. $\partial\nu = 0$ *for any scalar* ν.

2. $\partial\,\mathrm{tr}(\xi e) = \xi$ *for any numerical matrix* ξ.

3. (quantum Leibniz rule)

$$\partial(xy) = (\partial x)y + x(\partial y) + (\partial x)(\partial y) \tag{2.7}$$

for any elements x *and* y *of the tensor algebra* $Tgl(d,\mathbb{C})$.

Proof. 1. By definition.

2. We have

$$\partial^i_j e^{i_1}_{j_1} = \partial^i_j(1) e^{i_1}_{j_1} + \big(1\delta^i_{j_1} + \partial^i_{j_1}(1)\big)\delta^{i_1}_j = \delta^i_{j_1}\delta^{i_1}_j$$

by the equation (2.6) since we have $\partial^i_j(1) = 0$. We have $\partial^i_j \operatorname{tr}(\xi e) = \xi^i_j$ for any numerical matrix ξ.

3. We may assume that we have

$$x = e^{i_1}_{j_1} \cdots e^{i_m}_{j_m}, \qquad y = e^{i_{m+1}}_{j_{m+1}} \cdots e^{i_{m+n}}_{j_{m+n}}$$

since both sides of the equation (2.7) are complex bilinear. The proof is by induction on the nonnegative integer n. Suppose that we have $n > 0$. We have

$$\begin{aligned}\partial^i_j\Big(e^{i_1}_{j_1} \cdots e^{i_{m+n}}_{j_{m+n}}\Big) &= \partial^i_j\Big(e^{i_1}_{j_1} \cdots e^{i_{m+n-1}}_{j_{m+n-1}}\Big) e^{i_{m+n}}_{j_{m+n}} \\ &\quad + \Big(e^{i_1}_{j_1} \cdots e^{i_{m+n-1}}_{j_{m+n-1}} \delta^i_{j_{m+n}} + \partial^i_{j_{m+n}}\Big(e^{i_1}_{j_1} \cdots e^{i_{m+n-1}}_{j_{m+n-1}}\Big)\Big)\delta^{i_{m+n}}_j \end{aligned} \tag{2.8}$$

by definition. We have

$$\begin{aligned}\partial\Big(e^{i_1}_{j_1} \cdots e^{i_{m+n-1}}_{j_{m+n-1}}\Big) &= \partial\Big(e^{i_1}_{j_1} \cdots e^{i_m}_{j_m}\Big) e^{i_{m+1}}_{j_{m+1}} \cdots e^{i_{m+n-1}}_{j_{m+n-1}} \\ + e^{i_1}_{j_1} \cdots e^{i_m}_{j_m} \partial\Big(e^{i_{m+1}}_{j_{m+1}} \cdots e^{i_{m+n-1}}_{j_{m+n-1}}\Big) &+ \partial\Big(e^{i_1}_{j_1} \cdots e^{i_m}_{j_m}\Big)\partial\Big(e^{i_{m+1}}_{j_{m+1}} \cdots e^{i_{m+n-1}}_{j_{m+n-1}}\Big) \end{aligned} \tag{2.9}$$

by the induction hypothesis. We have

$$
\begin{aligned}
&\partial^i_j\left(e^{i_1}_{j_1}\cdots e^{i_{m+n}}_{j_{m+n}}\right)\\
&\quad= \partial^i_j\left(e^{i_1}_{j_1}\cdots e^{i_m}_{j_m}\right)e^{i_{m+1}}_{j_{m+1}}\cdots e^{i_{m+n}}_{j_{m+n}}\\
&\qquad\qquad\qquad + e^{i_1}_{j_1}\cdots e^{i_m}_{j_m}\Bigg(\partial^i_j\left(e^{i_{m+1}}_{j_{m+1}}\cdots e^{i_{m+n-1}}_{j_{m+n-1}}\right)e^{i_{m+n}}_{j_{m+n}}\\
&\qquad\quad + \left(e^{i_{m+1}}_{j_{m+1}}\cdots e^{i_{m+n-1}}_{j_{m+n-1}}\delta^i_{j_{m+n}} + \partial^i_{j_{m+n}}\left(e^{i_{m+1}}_{j_{m+1}}\cdots e^{i_{m+n-1}}_{j_{m+n-1}}\right)\right)\delta^{i_{m+n}}_j\Bigg)\\
&\qquad\qquad + \sum_{k=1}^{d}\partial^i_k\left(e^{i_1}_{j_1}\cdots e^{i_m}_{j_m}\right)\Bigg(\partial^k_j\left(e^{i_{m+1}}_{j_{m+1}}\cdots e^{i_{m+n-1}}_{j_{m+n-1}}\right)e^{i_{m+n}}_{j_{m+n}}\\
&\qquad\quad + \left(e^{i_{m+1}}_{j_{m+1}}\cdots e^{i_{m+n-1}}_{j_{m+n-1}}\delta^k_{j_{m+n}} + \partial^k_{j_{m+n}}\left(e^{i_{m+1}}_{j_{m+1}}\cdots e^{i_{m+n-1}}_{j_{m+n-1}}\right)\right)\delta^{i_{m+n}}_j\Bigg)\\
&\quad= \partial^i_j\left(e^{i_1}_{j_1}\cdots e^{i_m}_{j_m}\right)e^{i_{m+1}}_{j_{m+1}}\cdots e^{i_{m+n}}_{j_{m+n}} + e^{i_1}_{j_1}\cdots e^{i_m}_{j_m}\partial^i_j\left(e^{i_{m+1}}_{j_{m+1}}\cdots e^{i_{m+n}}_{j_{m+n}}\right)\\
&\qquad\qquad\qquad\qquad + \sum_{k=1}^{d}\partial^i_k\left(e^{i_1}_{j_1}\cdots e^{i_m}_{j_m}\right)\partial^k_j\left(e^{i_{m+1}}_{j_{m+1}}\cdots e^{i_{m+n}}_{j_{m+n}}\right)
\end{aligned}
$$

by the equations (2.8) and (2.9). □

Proposition 2.2.2. *The linear mapping* (2.5) *is the unique one satisfying the conditions of Proposition 2.2.1.*

Proposition 2.2.3. *The matrix elements of the linear mapping* (2.5) *commute with each other.*

Proof. The subspace

$$\bigcap_{i_1,j_1,i_2,j_2=1}^{d} \ker\left[\partial^{i_1}_{j_1},\partial^{i_2}_{j_2}\right] \tag{2.10}$$

contains the subspace $\mathbb{C}\oplus gl(d,\mathbb{C})$ and it is closed under multiplication. □

Proposition 2.2.4. *The matrix elements of the linear mapping* (2.5) *preserve the kernel of the canonical epimorphism of the tensor algebra onto the universal enveloping algebra.*

Proof. We have

$$\begin{aligned}
&\partial\Big(x\Big(e^{i_1}_{j_1}e^{i_2}_{j_2} - e^{i_2}_{j_2}e^{i_1}_{j_1} - e^{i_2}_{j_1}\delta^{i_1}_{j_2} + \delta^{i_2}_{j_1}e^{i_1}_{j_2}\Big)y\Big) \\
&\quad = (\partial x)\Big(e^{i_1}_{j_1}e^{i_2}_{j_2} - e^{i_2}_{j_2}e^{i_1}_{j_1} - e^{i_2}_{j_1}\delta^{i_1}_{j_2} + \delta^{i_2}_{j_1}e^{i_1}_{j_2}\Big)y \\
&\qquad + x\Big(e^{i_1}_{j_1}e^{i_2}_{j_2} - e^{i_2}_{j_2}e^{i_1}_{j_1} - e^{i_2}_{j_1}\delta^{i_1}_{j_2} + \delta^{i_2}_{j_1}e^{i_1}_{j_2}\Big)(\partial y) \\
&\qquad + (\partial x)\Big(e^{i_1}_{j_1}e^{i_2}_{j_2} - e^{i_2}_{j_2}e^{i_1}_{j_1} - e^{i_2}_{j_1}\delta^{i_1}_{j_2} + \delta^{i_2}_{j_1}e^{i_1}_{j_2}\Big)(\partial y)
\end{aligned}$$

for any elements x and y of the tensor algebra since we have

$$\begin{aligned}
&\partial\Big(e^{i_1}_{j_1}e^{i_2}_{j_2} - e^{i_2}_{j_2}e^{i_1}_{j_1} - e^{i_2}_{j_1}\delta^{i_1}_{j_2} + \delta^{i_2}_{j_1}e^{i_1}_{j_2}\Big) \\
&\quad = E^{i_1}_{j_1}e^{i_2}_{j_2} + e^{i_1}_{j_1}E^{i_2}_{j_2} - e^{i_2}_{j_2}E^{i_1}_{j_1} - E^{i_2}_{j_2}e^{i_1}_{j_1} \\
&\qquad + E^{i_1}_{j_1}E^{i_2}_{j_2} - E^{i_2}_{j_2}E^{i_1}_{j_1} - E^{i_2}_{j_1}\delta^{i_1}_{j_2} + \delta^{i_2}_{j_1}E^{i_1}_{j_2} \\
&\quad = \big[E^{i_1}_{j_1}, E^{i_2}_{j_2}\big] - E^{i_2}_{j_1}\delta^{i_1}_{j_2} + \delta^{i_2}_{j_1}E^{i_1}_{j_2} = 0.
\end{aligned}$$

□

Definition 2.2.2. The induced linear mapping

$$Ugl(d,\mathbb{C}) \to M\big(d, Ugl(d,\mathbb{C})\big), \qquad x \mapsto \partial x = (\partial^i_j x)^d_{i,j=1}$$

is called the quantum derivation of the universal enveloping algebra.

Theorem 2.2.1 ([15])**.** *The quantum derivation ∂ is the unique linear mapping satisfying the following.*

1. $\partial\nu = 0$ *for any scalar* ν.

2. $\partial\,\mathrm{tr}(\xi e) = \xi$ *for any numerical matrix* ξ.

3. *(quantum Leibniz rule)*

$$\partial(xy) = (\partial x)y + x(\partial y) + (\partial x)(\partial y)$$

for any elements x and y of the universal enveloping algebra $Ugl(d,\mathbb{C})$.

Proposition 2.2.5. *The matrix elements of the quantum derivation commute with each other.*

Proof. By Proposition 2.2.3. □

We write δ for the identity matrix of the matrix algebra

$$\hom\Big(Ugl(d,\mathbb{C}), M\big(d, Ugl(d,\mathbb{C})\big)\Big) = M\big(d, \hom Ugl(d,\mathbb{C})\big).$$

Theorem 2.2.2. *The mapping $\delta + \partial$ is a homomorphism of unital complex algebras.*

Proof. We have $(\delta + \partial)(1) = \delta$ and we have

$$\begin{aligned}(\delta + \partial)(xy) &= xy + \partial(xy)\\ &= xy + (\partial x)y + x(\partial y) + (\partial x)(\partial y)\\ &= (x + \partial x)(y + \partial y) = (\delta + \partial)(x)(\delta + \partial)(y)\end{aligned}$$

for any elements x and y of the universal enveloping algebra $Ugl(d,\mathbb{C})$. □

2.3 Another Construction

Another construction of the quantum derivation based on the fact that the mapping $\delta + \partial$ is a homomorphism of unital complex algebras is the following. The mapping

$$Tgl(d,\mathbb{C}) \to M\big(d, Ugl(d,\mathbb{C})\big), \qquad x \mapsto (\delta \circ \pi)(x) = \pi(x)\delta$$

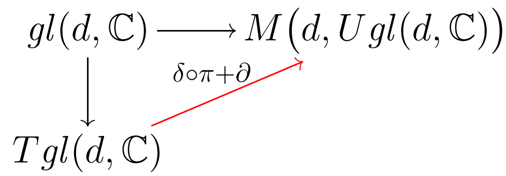
is a homomorphism of unital complex algebras. We define a linear mapping

$$gl(d,\mathbb{C}) \to M\big(d, Ugl(d,\mathbb{C})\big), \qquad \operatorname{tr}(\xi e) \mapsto \partial \operatorname{tr}(\xi e) = \xi \tag{2.11}$$

The linear mapping

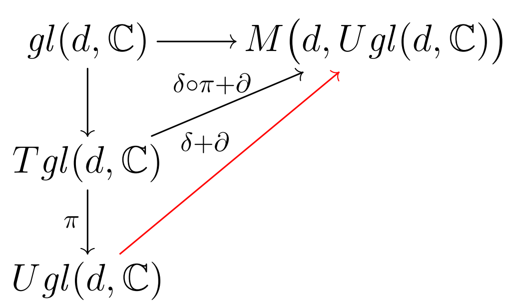
$$gl(d,\mathbb{C}) \to M\big(d, Ugl(d,\mathbb{C})\big), \qquad\qquad x \mapsto (\delta\circ\pi + \partial)(x)$$

extends uniquely to the homomorphism of unital complex algebras of the tensor algebra $Tgl(d,\mathbb{C})$.



The linear mapping ∂ factors uniquely through the canonical epimorphism π since we have

$$\begin{aligned}
&(\delta\circ\pi+\partial)\Big(e^{i_1}_{j_1}e^{i_2}_{j_2} - e^{i_2}_{j_2}e^{i_1}_{j_1} - e^{i_2}_{j_1}\delta^{i_1}_{j_2} + \delta^{i_2}_{j_1}e^{i_1}_{j_2}\Big)\\
&\quad= \Big((\delta\circ\pi+\partial)e^{i_1}_{j_1}\Big)\Big((\delta\circ\pi+\partial)e^{i_2}_{j_2}\Big)\\
&\qquad\qquad - \Big((\delta\circ\pi+\partial)e^{i_2}_{j_2}\Big)\Big((\delta\circ\pi+\partial)e^{i_1}_{j_1}\Big)\\
&\qquad\quad - \Big((\delta\circ\pi+\partial)e^{i_2}_{j_1}\Big)\delta^{i_1}_{j_2} + \delta^{i_2}_{j_1}\Big((\delta\circ\pi+\partial)e^{i_1}_{j_2}\Big)\\
&\quad= \Big(e^{i_1}_{j_1} + E^{i_1}_{j_1}\Big)\Big(e^{i_2}_{j_2} + E^{i_2}_{j_2}\Big) - \Big(e^{i_2}_{j_2} + E^{i_2}_{j_2}\Big)\Big(e^{i_1}_{j_1} + E^{i_1}_{j_1}\Big)\\
&\qquad\qquad\qquad - \Big(e^{i_2}_{j_1} + E^{i_2}_{j_1}\Big)\delta^{i_1}_{j_2} + \delta^{i_2}_{j_1}\Big(e^{i_1}_{j_2} + E^{i_1}_{j_2}\Big)\\
&\quad= e^{i_1}_{j_1}e^{i_2}_{j_2} - e^{i_2}_{j_2}e^{i_1}_{j_1} - e^{i_2}_{j_1}\delta^{i_1}_{j_2} + \delta^{i_2}_{j_1}e^{i_1}_{j_2}\\
&\qquad\quad + E^{i_1}_{j_1}E^{i_2}_{j_2} - E^{i_2}_{j_2}E^{i_1}_{j_1} - E^{i_2}_{j_1}\delta^{i_1}_{j_2} + \delta^{i_2}_{j_1}E^{i_1}_{j_2}\\
&\quad= 0.
\end{aligned}$$



The linear mapping ∂ is the quantum derivation.

1. We have $\nu + \partial\nu = (\delta + \partial)\nu = \nu$ for any scalar ν since the mapping $\delta + \partial$ is a homomorphism of unital complex algebras. We have $\partial\nu = 0$.

2. We have $\partial \operatorname{tr}(\xi e) = \xi$ for any numerical matrix ξ by the definition (2.11).

3. We have

$$\begin{aligned} xy + \partial(xy) &= (\delta + \partial)(xy) \\ &= (\delta + \partial)(x)(\delta + \partial)(y) \\ &= (x + \partial x)(y + \partial y) \\ &= xy + (\partial x)y + x(\partial y) + (\partial x)(\partial y) \end{aligned}$$

for any elements x and y of the universal enveloping algebra since the mapping $\delta + \partial$ is a homomorphism of unital complex algebras. We have the quantum Leibniz rule

$$\partial(xy) = (\partial x)y + x(\partial y) + (\partial x)(\partial y).$$

2.4 Key Formula for the Quantum Derivation

The center of the universal enveloping algebra is freely generated by the finite sequence $\left(\operatorname{tr} e, \dots, \operatorname{tr} e^d\right)$ and we have the isomorphism

$$\mathbb{C}\left[x_1, \dots, x_d\right] = \mathbb{C}\left[\operatorname{tr} e, \dots, \operatorname{tr} e^d\right] = C.$$

Therefore, it is important, and even sufficient, to obtain the formula for the quantum derivations of the matrix elements $(e^n)^i_j$ in order to calculate higher order quantum argument shifts $\partial_\xi x$ for central elements x. We begin with the commutation relations of these matrix elements.

Proposition 2.4.1. *We have*

$$\left[(e^n)^{i_1}_{j_1}, e^{i_2}_{j_2}\right] = \left[e^{i_1}_{j_1}, (e^n)^{i_2}_{j_2}\right] = (e^n)^{i_2}_{j_1}\delta^{i_1}_{j_2} - \delta^{i_2}_{j_1}(e^n)^{i_1}_{j_2}$$

for any nonnegative integer n.

Proof. The proof is by induction on the nonnegative integer n. Suppose that we have $n > 1$. We have

$$\begin{aligned}
\left[(e^n)^{i_1}_{j_1}, e^{i_2}_{j_2}\right] &= \sum_{k=1}^{d} \left[(e^{n-1})^{i_1}_k e^k_{j_1}, e^{i_2}_{j_2}\right] \\
&= \sum_{k=1}^{d} \Big(\left[(e^{n-1})^{i_1}_k, e^{i_2}_{j_2}\right] e^k_{j_1} + (e^{n-1})^{i_1}_k \left[e^k_{j_1}, e^{i_2}_{j_2}\right] \Big) \\
&= \sum_{k=1}^{d} \bigg(\Big((e^{n-1})^{i_2}_k \delta^{i_1}_{j_2} - \delta^{i_2}_k (e^{n-1})^{i_1}_{j_2} \Big) e^k_{j_1} + (e^{n-1})^{i_1}_k \Big(e^{i_2}_{j_1} \delta^k_{j_2} - \delta^{i_2}_{j_1} e^k_{j_2} \Big) \bigg) \\
&= (e^n)^{i_2}_{j_1} \delta^{i_1}_{j_2} - \delta^{i_2}_{j_1} (e^n)^{i_1}_{j_2}
\end{aligned}$$

by the induction hypothesis. □

The set $\left\{\operatorname{tr} e^n\right\}_{n=0}^{\infty}$ is contained in the center of the universal enveloping algebra since we have

$$\begin{aligned}
\left[\operatorname{tr} e^n, e\right] &= \sum_{k=1}^{d} \left[(e^n)^k_k, e\right] \\
&= \sum_{k=1}^{d} \Big((e^n)_k \delta^k - \delta_k (e^n)^k \Big) \\
&= e^n - e^n = 0
\end{aligned}$$

by Proposition 2.4.1.

Next we define the polynomials which are necessary to describe the quantum derivations $\partial (e^n)^i_j$.

Definition 2.4.1. We define the polynomials $g^{(n)}_m(x)$ and $h^{(n)}_m(x)$ by

1. $g^{(n)}_m(x) = g^{(n-1)}_m(x)x + h^{(n-1)}_m(x)$ for $0 \leq m < n$.

2. $g^{(n)}_n(x) = 1$ for $0 \leq n$.

3. $h_0^{(n)}(x) = \sum_{m=0}^{n-1} g_m^{(n-1)}(x) x^m$ for $0 \leq n$.

4. $h_m^{(n)}(x) = h_{m-1}^{(n-1)}(x)$ for $0 < m \leq n$.

We have the following.

Proposition 2.4.2. *The quantum derivations of the matrix elements* $(e^{n+1})^i_j$ *are given by the following formula*

$$\partial (e^{n+1})^i_j = \sum_{m=0}^{n} \Big(g_m^{(n)}(e)_j (e^m)^i + h_m^{(n)}(e)(e^m)^i_j \Big)$$

for any nonnegative integer n.

Proof. The proof is by induction on the nonnegative integer n. We have

$$\partial e^i_j = \delta_j \delta^i = g_0^{(0)}(e)_j \delta^i + h_0^{(0)}(e) \delta^i_j.$$

Suppose that we have $n > 0$. We have

$$\begin{aligned}
\partial (e^{n+1})^i_j &= \sum_{k=1}^{d} \partial \Big((e^n)^i_k e^k_j \Big) \\
&= \sum_{k=1}^{d} \Big(\partial (e^n)^i_k e^k_j + (e^n)^i_k \partial e^k_j + \partial (e^n)^i_k \partial e^k_j \Big) \\
&= \sum_{k=1}^{d} \partial (e^n)^i_k e^k_j + \delta_j (e^n)^i + \sum_{k=1}^{d} \partial (e^n)^i_k \delta_j \delta^k
\end{aligned}$$

by the quantum Leibniz rule. We have

$$\partial (e^n)^i_k = \sum_{m=0}^{n-1} \Big(g_m^{(n-1)}(e)_k (e^m)^i + h_m^{(n-1)}(e)(e^m)^i_k \Big)$$

by the induction hypothesis. We have

$$
\begin{aligned}
&\sum_{k=1}^{d}\sum_{m=0}^{n-1} g_m^{(n-1)}(e)_k (e^m)^i e_j^k - \sum_{m=0}^{n-1}\Big(g_m^{(n-1)}(e)e\Big)_j (e^m)^i \\
&\quad = \sum_{k=1}^{d}\sum_{m=0}^{n-1} g_m^{(n-1)}(e)_k \big[(e^m)^i, e_j^k\big] \\
&\quad = \sum_{k=1}^{d}\sum_{m=0}^{n-1} g_m^{(n-1)}(e)_k \Big((e^m)^k \delta_j^i - \delta^k (e^m)_j^i\Big) \\
&\quad = \sum_{m=0}^{n-1}\Big(g_m^{(n-1)}(e) e^m \delta_j^i - g_m^{(n-1)}(e)(e^m)_j^i\Big) \qquad (2.12)
\end{aligned}
$$

by Proposition 2.4.1. We have

$$
\sum_{k=1}^{d}\sum_{m=0}^{n-1} h_m^{(n-1)}(e)(e^m)_k^i e_j^k = \sum_{m=0}^{n-1} h_m^{(n-1)}(e)(e^{m+1})_j^i.
$$

We have

$$
\begin{aligned}
&\sum_{k=1}^{d}\sum_{m=0}^{n-1}\Big(g_m^{(n-1)}(e)_k (e^m)^i + h_m^{(n-1)}(e)(e^m)_k^i\Big)\delta_j \delta^k \\
&\qquad = \sum_{m=0}^{n-1}\Big(g_m^{(n-1)}(e)(e^m)_j^i + h_m^{(n-1)}(e)_j (e^m)^i\Big). \qquad (2.13)
\end{aligned}
$$

The second term of the equation (2.12) and the first term of the equation (2.13) are cancelled out and we have

$$\begin{aligned}
\partial(e^{n+1})^i_j = & \sum_{m=0}^{n-1}\Big(g_m^{(n-1)}(e)e + h_m^{(n-1)}(e)\Big)_j (e^m)^i + \delta_j (e^n)^i \\
& \quad + \sum_{m=0}^{n-1}\Big(g_m^{(n-1)}(e)e^m\delta^i_j + h_m^{(n-1)}(e)(e^{m+1})^i_j\Big) \\
= & \sum_{m=0}^{n-1}\Big(g_m^{(n-1)}(e)e + h_m^{(n-1)}(e)\Big)_j (e^m)^i + \delta_j (e^n)^i \\
& \quad + \sum_{m=0}^{n-1} g_m^{(n-1)}(e)e^m\delta^i_j + \sum_{m=1}^{n} h_{m-1}^{(n-1)}(e)(e^m)^i_j \\
= & \sum_{m=0}^{n-1} g_m^{(n)}(e)_j (e^m)^i + g_n^{(n)}(e)_j (e^n)^i \\
& \quad + h_0^{(n)}(e)\delta^i_j + \sum_{m=1}^{n} h_m^{(n)}(e)(e^m)^i_j \\
= & \sum_{m=0}^{n}\Big(g_m^{(n)}(e)_j (e^m)^i + h_m^{(n)}(e)(e^m)^i_j\Big).
\end{aligned}$$

□

Definition 2.4.2. We define

$$f_{\pm}^{(n)}(x) = \frac{(x+1)^n \pm (x-1)^n}{2} = \sum_{m=0}^{n} \frac{1 \pm (-1)^{n-m}}{2}\binom{n}{m} x^m$$

for any nonnegative integer n.

Lemma 2.4.1. *We have*

$$f_{\pm}^{(n+1)}(x) = f_{\pm}^{(n)}(x)x + f_{\mp}^{(n)}(x)$$

for any nonnegative integer n.

Proof. We have

$$\begin{aligned} f_{\pm}^{(n+1)}(x) &= \frac{(x+1)^{n+1} \pm (x-1)^{n+1}}{2} \\ &= \frac{(x+1)^n x + (x+1)^n}{2} \pm \frac{(x-1)^n x - (x-1)^n}{2} \\ &= \frac{(x+1)^n \pm (x-1)^n}{2} x + \frac{(x+1)^n \mp (x-1)^n}{2} \\ &= f_{\pm}^{(n)}(x)x + f_{\mp}^{(n)}(x). \end{aligned}$$

□

Lemma 2.4.2. *We have*

$$f_{-}^{(n+1)}(x) = \sum_{m=0}^{n} f_{+}^{(n-m)}(x)x^m$$

for any nonnegative integer n.

Proof. The proof is by induction on the nonnegative integer n. Suppose that we have $n > 0$. We have $f_{-}^{(n+1)}(x) = f_{-}^{(n)}(x)x + f_{+}^{(n)}(x)$ by Lemma 2.4.1. We have

$$\begin{aligned} f_{-}^{(n)}(x)x + f_{+}^{(n)}(x) &= \sum_{m=0}^{n-1} f_{+}^{(n-m-1)}(x)x^{m+1} + f_{+}^{(n)}(x) \\ &= \sum_{m=1}^{n} f_{+}^{(n-m)}(x)x^m + f_{+}^{(n)}(x) \\ &= \sum_{m=0}^{n} f_{+}^{(n-m)}(x)x^m \end{aligned}$$

by the induction hypothesis. □

Theorem 2.4.1 ([17])**.** *The quantum derivations of the matrix elements $(e^n)^i_j$ are given by the following formulae*

$$\begin{aligned} \partial(e^n)^i_j &= \sum_{m=0}^{n-1} \Big(f_{+}^{(n-m-1)}(e)_j (e^m)^i + f_{-}^{(n-m-1)}(e)(e^m)^i_j \Big) \\ &= \sum_{m=0}^{n-1} \Big((e^m)_j f_{+}^{(n-m-1)}(e)^i + (e^m)^i_j f_{-}^{(n-m-1)}(e) \Big) \qquad (2.14) \end{aligned}$$

for any nonnegative integer n.

Proof. The proof is by induction on the nonnegative integer n. Suppose that we have $n > 0$. We have

$$\begin{aligned}
\partial(e^n)^i_j &= \sum_{k=1}^{d} \partial\Big((e^{n-1})^i_k e^k_j\Big) \\
&= \sum_{k=1}^{d} \Big(\partial(e^{n-1})^i_k e^k_j + (e^{n-1})^i_k \partial e^k_j + \partial(e^{n-1})^i_k \partial e^k_j\Big) \\
&= \sum_{k=1}^{d} \partial(e^{n-1})^i_k e^k_j + \delta_j (e^{n-1})^i + \sum_{k=1}^{d} \partial(e^{n-1})^i_k \delta_j \delta^k
\end{aligned}$$

by the quantum Leibniz rule. We have

$$\partial(e^{n-1})^i_k = \sum_{m=0}^{n-2} \Big(f_+^{(n-m-2)}(e)_k (e^m)^i + f_-^{(n-m-2)}(e)(e^m)^i_k\Big)$$

by the induction hypothesis. We have

$$\begin{aligned}
\sum_{k=1}^{d}\sum_{m=0}^{n-2} & f_+^{(n-m-2)}(e)_k (e^m)^i e^k_j - \sum_{m=0}^{n-2} \Big(f_+^{(n-m-2)}(e)e\Big)_j (e^m)^i \\
&= \sum_{k=1}^{d}\sum_{m=0}^{n-2} f_+^{(n-m-2)}(e)_k \big[(e^m)^i, e^k_j\big] \\
&= \sum_{k=1}^{d}\sum_{m=0}^{n-2} f_+^{(n-m-2)}(e)_k \Big((e^m)^k \delta^i_j - \delta^k (e^m)^i_j\Big) \\
&= \sum_{m=0}^{n-2} \Big(f_+^{(n-m-2)}(e) e^m \delta^i_j - f_+^{(n-m-2)}(e)(e^m)^i_j\Big) \qquad (2.15)
\end{aligned}$$

by Proposition 2.4.1. We have

$$\sum_{k=1}^{d}\sum_{m=0}^{n-2} f_-^{(n-m-2)}(e)(e^m)^i_k e^k_j = \sum_{m=0}^{n-2} f_-^{(n-m-2)}(e)(e^{m+1})^i_j.$$

We have

$$\sum_{k=1}^{d}\sum_{m=0}^{n-2}\Big(f_+^{(n-m-2)}(e)_k(e^m)^i + f_-^{(n-m-2)}(e)(e^m)^i_k\Big)\delta_j\delta^k$$
$$= \sum_{m=0}^{n-2}\Big(f_+^{(n-m-2)}(e)(e^m)^i_j + f_-^{(n-m-2)}(e)_j(e^m)^i\Big). \quad (2.16)$$

The second term of the equation (4.2) and the first term of the equation (2.16) are cancelled out and we have

$$\begin{aligned}\partial(e^n)^i_j &= \sum_{m=0}^{n-2}\Big(f_+^{(n-m-2)}(e)e + f_-^{(n-m-2)}(e)\Big)_j(e^m)^i + \delta_j(e^{n-1})^i \\ &\quad + \sum_{m=0}^{n-2}\Big(f_+^{(n-m-2)}(e)e^m\delta^i_j + f_-^{(n-m-2)}(e)(e^{m+1})^i_j\Big) \\ &= \sum_{m=0}^{n-2} f_+^{(n-m-1)}(e)_j(e^m)^i + \delta_j(e^{n-1})^i \\ &\quad + f_-^{(n-1)}(e)\delta^i_j + \sum_{m=1}^{n-1} f_-^{(n-m-1)}(e)(e^m)^i_j \\ &= \sum_{m=0}^{n-1}\Big(f_+^{(n-m-1)}(e)_j(e^m)^i + f_-^{(n-m-1)}(e)(e^m)^i_j\Big)\end{aligned}$$

by Lemma 2.4.1 and 2.4.2. We similarly prove the equation (2.14). □

2.5 Main Theorem for $m = n = 1$

In this section, we apply Theorem 2.4.1 to show that the quantum argument shifts of central elements of the universal enveloping algebra $Ugl(d, \mathbb{C})$ commute with each other. We adopt the convention $\operatorname{tr} e^{-1} = 1$.

Definition 2.5.1. We define the polynomials

$$f_\pm^{(n)}(x) = \sum_{m=0}^{n+1}\frac{1 \pm (-1)^{n-m}}{2}\binom{n}{m}x^m$$

consistently with Definition 2.4.2.

Remark 2.5.1. We have

$$f_+^{(-1)}(x) = 0, \qquad\qquad f_-^{(-1)}(x) = \binom{-1}{0} = 1.$$

Using this notation, we have

$$\begin{aligned}
\partial \operatorname{tr} e^n &= \sum_{m=0}^{n-1} \Big(\sum_{k=1}^{d} f_+^{(n-m-1)}(e)_k (e^m)^k + f_-^{(n-m-1)}(e) \operatorname{tr} e^m \Big) \\
&= \sum_{m=0}^{n-1} \Big(f_+^{(n-m-1)}(e) e^m + f_-^{(n-m-1)}(e) \operatorname{tr} e^m \Big) \\
&= f_-^{(n)}(e) + \sum_{m=0}^{n-1} f_-^{(n-m-1)}(e) \operatorname{tr} e^m
\end{aligned}$$

by Lemma 2.4.2 and

$$\begin{aligned}
(\delta + \partial) \operatorname{tr} e^n &= \operatorname{tr} e^n + \sum_{m=-1}^{n-1} f_-^{(n-m-1)}(e) \operatorname{tr} e^m \\
&= \sum_{m=-1}^{n} f_-^{(n-m-1)}(e) \operatorname{tr} e^m
\end{aligned} \tag{2.17}$$

by the convention $\operatorname{tr} e^{-1} = 1$. Recall that the center of the universal enveloping algebra $U gl(d, \mathbb{C})$ is generated by the elements $\operatorname{tr} e^n$. We have the following.

Theorem 2.5.1 ([19])**.**

$$(\delta + \partial) \prod_m \operatorname{tr} e^{n_m} = \sum_{m_1=-1}^{n_1} \operatorname{tr} e^{m_1} \sum_{m_2=-1}^{n_2} \operatorname{tr} e^{m_2} \cdots \prod_k f_-^{(n_k - m_k - 1)}(e)$$

for a finite product $\prod_m \operatorname{tr} e^{n_m}$.

Proof. We have

$$\begin{aligned}(\delta+\partial)\prod_m \operatorname{tr} e^{n_m} &= \prod_k (\delta+\partial)\operatorname{tr} e^{n_k} \\ &= \prod_k \sum_{m_k=-1}^{n_k} f_-^{(n_k-m_k-1)}(e)\operatorname{tr} e^{m_k} \\ &= \sum_{m_1=-1}^{n_1} \operatorname{tr} e^{m_1} \sum_{m_2=-1}^{n_2} \operatorname{tr} e^{m_2}\cdots \prod_k f_-^{(n_k-m_k-1)}(e)\end{aligned}$$

by Theorem 2.2.2 and the equation (2.17). □

Suppose that ξ is an arbitrary numerical matrix.

Definition 2.5.2. The mapping $\partial_\xi = \operatorname{tr}(\xi\partial)$ is called the quantum argument shift operator in the direction ξ.

We obtained the quantum argument shift in the direction ξ of any central element.

Corollary 2.5.1 ([19])**.**

$$(\operatorname{tr}\xi+\partial_\xi)\prod_m \operatorname{tr} e^{n_m} = \sum_{m_1=-1}^{n_1} \operatorname{tr} e^{m_1} \sum_{m_2=-1}^{n_2} \operatorname{tr} e^{m_2}\cdots \operatorname{tr}\Big(\xi \prod_k f_-^{(n_k-m_k-1)}(e)\Big)$$

for a finite product $\prod_m \operatorname{tr} e^{n_m}$.

As a consequence, the set $\partial_\xi C$ (where C is the center of the universal enveloping algebra $U gl(d,\mathbb{C})$) is contained in the C-module

$$\operatorname{span}_C\Big\{ \operatorname{tr}\big(\xi e^n\big) : n = 0,1,2,\dots \Big\}.$$

Suppose that $\big(\zeta_1,\dots,\zeta_n\big)$ is a finite sequence of the set $M(d,\mathbb{C}) \sqcup \{e\}$.

1. We have

$$\Big[\operatorname{tr}\big(\xi e^m\big), \operatorname{tr}\big(\zeta_1\cdots\zeta_n\big)\Big] = \sum_{\zeta_k=e} \operatorname{tr}\Big(\zeta_1\cdots\zeta_{k-1}\big[e^m,\xi\big]\zeta_{k+1}\cdots\zeta_n\Big) \tag{2.18}$$

since we have

$$\Big[\operatorname{tr}\big(\xi e^m\big), e\Big] = \sum_{i,j=1}^{d} \xi_i^j \big[(e^m)_j^i, e\big]$$
$$= \sum_{i,j=1}^{d} \xi_i^j \Big((e^m)_j \delta^i - \delta_j (e^m)^i\Big) = \big[e^m, \xi\big]$$

by Proposition 2.4.1.

2. We have

$$\operatorname{tr}\big[\xi e^m, \zeta_1 \cdots \zeta_n\big] = \sum_{\zeta_k = e} \Big(\operatorname{tr}\big(\zeta_1 \cdots \zeta_{k-1} e^m\big) \operatorname{tr}\big(\xi \zeta_{k+1} \cdots \zeta_n\big) - \operatorname{tr}\big(\zeta_1 \cdots \zeta_{k-1}\big) \operatorname{tr}\big(\xi e^m \zeta_{k+1} \cdots \zeta_n\big)\Big) \quad (2.19)$$

by Proposition 2.4.1.

Proposition 2.5.1. *We have*

$$\Big[\operatorname{tr}\big(\xi e^m\big), \operatorname{tr}\big(\xi e^n\big)\Big] = \operatorname{tr}\big[\xi e^m, \xi e^n\big] = 0$$

for any nonnegative integers m and n.

Proof. The proof is by induction on the nonnegative integer $m+n$. Suppose that we have $m+n>0$. We have

$$\Big[\operatorname{tr}\big(\xi e^m\big), \operatorname{tr}\big(\xi e^n\big)\Big] = \sum_{k=1}^{n} \operatorname{tr}\Big(\xi e^{k-1}\big[e^m, \xi\big] e^{n-k}\Big)$$
$$= \sum_{k=1}^{n} \Big(\operatorname{tr}\big(\xi e^{m+k-1} \xi e^{n-k}\big) - \operatorname{tr}\big(\xi e^{k-1} \xi e^{m+n-k}\big)\Big)$$
$$= \sum_{k=1}^{n} \operatorname{tr}\big[\xi e^{m+k-1}, \xi e^{n-k}\big] = 0$$

by the equation (2.18) and

$$\begin{aligned}\operatorname{tr}\bigl[\xi e^m, \xi e^n\bigr] &= \sum_{k=1}^{n}\Bigl(\operatorname{tr}\bigl(\xi e^{m+k-1}\bigr)\operatorname{tr}\bigl(\xi e^{n-k}\bigr) - \operatorname{tr}\bigl(\xi e^{k-1}\bigr)\operatorname{tr}\bigl(\xi e^{m+n-k}\bigr)\Bigr)\\ &= \sum_{k=1}^{n}\Bigl[\operatorname{tr}\bigl(\xi e^{m+k-1}\bigr), \operatorname{tr}\bigl(\xi e^{n-k}\bigr)\Bigr] = 0\end{aligned}$$

by the equation (2.19). □

We obtained the main theorem (Theorem 3.1.1) for $m = n = 1$.

Theorem 2.5.2 ([17]). *We have*

$$[\partial_\xi x, \partial_\xi y] = 0$$

for any central elements x and y of the universal enveloping algebra $Ugl(d, \mathbb{C})$.

We define $C_\xi^{(0)} = C$ and let $C_\xi^{(n)}$ be the algebra generated by the algebra $C_\xi^{(n-1)}$ and the vector space $\partial_\xi^n C$. We define

$$C_\xi = \lim_{n\to\infty} C_\xi^{(n)} = \bigcup_{n=0}^{\infty} C_\xi^{(n)}.$$

We identified the generators of the algebra $C_\xi^{(1)}$. We write $C\bigl[x_n : n = 1, 2, \ldots\bigr]$ for the algebra generated by the center C and a subset $\bigl\{\, x_n : n = 1, 2, \ldots \bigr\}$ of the universal enveloping algebra $Ugl(d, \mathbb{C})$. Here, the elements $\bigl\{\, x_n : n = 1, 2, \ldots \bigr\}$ are assumted to commute with each other, but they could be algebraically dependent over the center C.

Theorem 2.5.3 ([17]).

$$C_\xi^{(1)} = C\Bigl[\, \operatorname{tr}\bigl(\xi e^n\bigr) : n = 1, 2, \ldots \Bigr] = C\Bigl[\, \partial_\xi \operatorname{tr} e^n : n = 2, 3, \ldots \Bigr].$$

Proof. The algebra $C_\xi^{(1)}$ is contained in the algebra

$$C\Bigl[\, \operatorname{tr}\bigl(\xi e^n\bigr) : n = 1, 2, \ldots \Bigr]$$

by Corollary 2.5.1. We prove that the elements $\operatorname{tr}\bigl(\xi e^n\bigr)$ belong to the algebra

$$C\Bigl[\,\partial_\xi \operatorname{tr} e^n : n = 2, 3, \dots\Bigr] \tag{2.20}$$

by induction on the nonnegative integer n. Suppose that the integer n is positive and the element $\operatorname{tr}\bigl(\xi e^m\bigr)$ belongs to the algebra 2.20 for any nonnegative integer $m < n$. The element $\operatorname{tr}\bigl(\xi e^n\bigr)$ belongs to the algebra (2.20) since the element

$$\begin{aligned}(\operatorname{tr}\xi + \partial_\xi) \operatorname{tr} e^{n+1} - (n+1) \operatorname{tr}\bigl(\xi e^n\bigr) \\ = \sum_{m=-1}^{n+1} \operatorname{tr} e^m \operatorname{tr}\bigl(\xi f_-^{(n-m)}(e)\bigr) - (n+1) \operatorname{tr}\bigl(\xi e^n\bigr)\end{aligned}$$

belongs to the submodule $\operatorname{span}_C\Bigl\{\operatorname{tr}\bigl(\xi e^m\bigr)\Bigr\}_{m=0}^{n-1}$. □

Chapter 3

Quantum Analog of Mishchenko-Fomenko Theorem

3.1 Introduction

In Theorem 1.2.2 in Chapter 1, I explained that for a finite dimensional complex Lie algebra g with a basis $(e_1, \dots, e_d)$, the Poisson algebra $(Sg, \{,\})$ (where $\{,\}$ is a Lie–Poisson bracket on Sg) and a derivation

$$\bar{\partial}_\xi = \xi(e_1)\frac{\partial}{\partial e_1} + \cdots + \xi(e_d)\frac{\partial}{\partial e_d}$$

(where $\bar{\partial}_\xi$ is a linear operator on $Sg = \mathbb{C}[e_1, \dots, e_d]$) in the direction $\xi \in g^*$ satisfy the argument shift method

$$\{\bar{\partial}_\xi^m x, \bar{\partial}_\xi^n y\} = 0 \tag{3.1}$$

for Poisson central elements x and y of Sg. As a certain quantisation of the Poisson algebra $(Sg, \{,\})$, we can consider the universal enveloping algebra $(Ug, [,])$

(where $[,]$ is a commutation relation on Ug). We can define the quantum analog of the relation (3.1) once we define the quantisation ∂_ξ (a linear operator on Ug) of the derivation $\bar{\partial}_\xi$. At the moment, it seems clearly effective to define the quantum version ∂ of the partial differential operator

$$\bar{\partial} = \left(\frac{\partial}{\partial e_1}, \ldots, \frac{\partial}{\partial e_d} \right).$$

However, since Ug is noncommutative due to the Lie bracket of the Lie algebra g, we cannot expect that the quantum version ∂ will satisfy the Leibniz rule (see equation (2.3)). As I explained in Chapter 2, Gurevich, Pyatov, and Saponov [15] noticed that when $g = gl(d, \mathbb{C})$ they could modify the Leibniz rule:

$$\partial(xy) = (\partial x)y + x(\partial y) + (\partial x)(\partial y),$$

where $(\partial x)(\partial y)$ is a product of the matrices ∂x and $\partial y \in M\big(d, Ugl(d, \mathbb{C})\big)$. Using a quantum derivation ∂ defined as above (see Theorem 2.2.1), we can formulate the quantum analog of the theorem of Mishchenko and Fomenko:

Theorem 3.1.1 ([18]). *Suppose that ξ is an arbitrary element of the dual space $gl(d, \mathbb{C})^*$ and let*

$$\partial_\xi = \operatorname{tr}(\xi \partial) = \sum_{i,j=1}^{d} \xi(e^i_j) \partial^j_i.$$

We have

$$[\partial^m_\xi x, \partial^n_\xi y] = 0$$

for any positive integers m and n and for any central elements x and y of the universal enveloping algebra $Ugl(d, \mathbb{C})$.

I will explain our proof of the theorem in this chapter.

3.2 Adjoint Action on Quantum Argument Shifts

I will explain the adjoint action of the Lie group $GL(d,\mathbb{C})$ on the quantum argument shifts ∂_ξ, which is needed to reduce Theorem 3.1.1 to the case where the matrix $\xi \in gl(d,\mathbb{C})^* \simeq M(d,\mathbb{C})$ is diagonal and has d distinct eigenvalues. We use the following identification.

$$gl(d,\mathbb{C}) = M(d,\mathbb{C}), \qquad \operatorname{tr}(\xi e) = \begin{pmatrix} \xi_1^1 & \dots & \xi_d^1 \\ \multicolumn{3}{c}{\dots\dots\dots\dots} \\ \xi_1^d & \dots & \xi_d^d \end{pmatrix},$$

$$GL(d,\mathbb{C}) = M(d,\mathbb{C})^\times, \qquad g = \begin{pmatrix} g_1^1 & \dots & g_d^1 \\ \multicolumn{3}{c}{\dots\dots\dots\dots} \\ g_1^d & \dots & g_d^d \end{pmatrix},$$

$$gl(d,\mathbb{C})^* = M(d,\mathbb{C}), \qquad \xi = \begin{pmatrix} \xi(e_1^1) & \dots & \xi(e_d^1) \\ \multicolumn{3}{c}{\dots\dots\dots\dots\dots\dots} \\ \xi(e_1^d) & \dots & \xi(e_d^d) \end{pmatrix}.$$

We consider the adjoint action of the Lie group $GL(d,\mathbb{C})$. Suppose that g is an element of the Lie group $GL(d,\mathbb{C})$. The automorphism $\operatorname{Ad} g$ extends uniquely to the automorphism of the universal enveloping algebra $U gl(d,\mathbb{C})$ and the tensor product $\operatorname{Ad} g \otimes \delta$ is an automorphism of the matrix algebra $U gl(d,\mathbb{C}) \otimes M(d,\mathbb{C}) = M\big(d, U gl(d,\mathbb{C})\big)$. We have

$$(\operatorname{Ad} g \otimes \delta) e = \begin{pmatrix} (\operatorname{Ad} g) e_1^1 & \dots & (\operatorname{Ad} g) e_d^1 \\ \multicolumn{3}{c}{\dots\dots\dots\dots\dots\dots\dots\dots} \\ (\operatorname{Ad} g) e_1^d & \dots & (\operatorname{Ad} g) e_d^d \end{pmatrix} = g^{-1} e g$$

since for an invertible matrix g we have

$$
\begin{aligned}
(\operatorname{Ad} g)E^i_j &= gE^i_j g^{-1} \\
&= g_j (g^{-1})^i \\
&= \sum_{i',j'=1}^{d} g^{j'}_j (g^{-1})^i_{i'} E^{i'}_{j'} \\
&= \sum_{i',j'=1}^{d} (g^{-1})^i_{i'} E^{i'}_{j'} g^{j'}_j .
\end{aligned}
$$

The matrix $g^{-1}eg$ is another generating matrix:

1. The matrix elements of the matrix $g^{-1}eg$ form a basis of the Lie algebra $gl(d, \mathbb{C})$.

2. We have the commutation relation

$$
\big[(g^{-1}eg)^{i_1}_{j_1}, (g^{-1}eg)^{i_2}_{j_2}\big] = \delta^{i_1}_{j_2}(g^{-1}eg)^{i_2}_{j_1} - (g^{-1}eg)^{i_1}_{j_2}\delta^{i_2}_{j_1}.
$$

Proposition 3.2.1. *We have the following.*

1. *The following diagram commutes.*

$$
\begin{array}{ccc}
M\big(d, U\,gl(d,\mathbb{C})\big) & \xrightarrow{\operatorname{Ad} g \otimes \delta} & M\big(d, U\,gl(d,\mathbb{C})\big) \\
\partial \Big\uparrow & & \Big\uparrow g^{-1}\partial g \\
U\,gl(d,\mathbb{C}) & \xrightarrow{\operatorname{Ad} g} & U\,gl(d,\mathbb{C})
\end{array}
\tag{3.2}
$$

2. *The linear mapping $g^{-1}\partial g$ is the quantum derivation of the universal enveloping algebra with respect to the generating matrix $(\operatorname{Ad} g \otimes \delta)e = g^{-1}eg$:*

   (a) $(g^{-1}\partial g)\nu = 0$ for any scalar ν.

   (b) $(g^{-1}\partial g)\,\operatorname{tr}(\xi g^{-1}eg) = \xi$ for any numerical matrix ξ.

(c) (quantum Leibniz rule)

$$(g^{-1}\partial g)(xy) = (g^{-1}\partial g)(x)y + x(g^{-1}\partial g)(y) + (g^{-1}\partial g)(x)(g^{-1}\partial g)(y)$$

for any elements x and y of the universal enveloping algebra $U gl(d, \mathbb{C})$.

Proof. The linear mapping

$$(\operatorname{Ad} g \otimes \delta) \circ \partial \circ \operatorname{Ad} g^{-1}$$

is the quantum derivation of the universal enveloping algebra with respect to the generating matrix $(\operatorname{Ad} g \otimes \delta)e = g^{-1}eg$. The linear mapping $g^{-1}\partial g$ is also the quantum derivation of the universal enveloping algebra with respect to the generating matrix $(\operatorname{Ad} g \otimes \delta)e = g^{-1}eg$:

1. We have $(g^{-1}\partial g)\nu = g^{-1}(\partial \nu)g = 0$ for any scalar ν.

2. We have

$$\begin{aligned}(g^{-1}\partial g)\operatorname{tr}(\xi g^{-1}eg) &= g^{-1}\partial \operatorname{tr}(\xi g^{-1}eg)g \\ &= g^{-1}g\xi g^{-1}g = \xi.\end{aligned}$$

   for any numerical matrix ξ.

3. We have

$$\begin{aligned}(g^{-1}\partial g)(xy) &= g^{-1}\partial(xy)g \\ &= g^{-1}\big((\partial x)y + x(\partial y) + (\partial x)(\partial y)\big)g \\ &= g^{-1}(\partial x)gy + xg^{-1}(\partial y)g + g^{-1}(\partial x)gg^{-1}(\partial y)g \\ &= (g^{-1}\partial g)(x)y + x(g^{-1}\partial g)(y) + (g^{-1}\partial g)(x)(g^{-1}\partial g)(y)\end{aligned}$$

   for any elements x and y of the universal enveloping algebra.

The diagram (3.2) commutes by Theorem 2.2.1 since both the linear mappings $(\operatorname{Ad} g \otimes \delta) \circ \partial \circ \operatorname{Ad} g^{-1}$ and $g^{-1}\partial g$ are the quantum derivation of the universal enveloping algebra with respect to the generating matrix $(\operatorname{Ad} g \otimes \delta)e = g^{-1}eg$. □

Suppose that ξ is an element of the dual space $gl(d, \mathbb{C})^*$.

Definition 3.2.1. We define

$$\partial_\xi = \sum_{i,j=1}^{d} \xi(e^i_j)\partial^j_i = \operatorname{tr}(\xi\partial),$$

$$(g^{-1}\partial g)_\xi = \sum_{i,j=1}^{d} \xi\big((g^{-1}eg)^i_j\big)(g^{-1}\partial g)^j_i.$$

Proposition 3.2.2. *We have* $\partial_\xi = (g^{-1}\partial g)_\xi$.

Proof. We have

$$\begin{aligned}(g^{-1}\partial g)_\xi &= \sum_{i,j=1}^{d} \xi\big((g^{-1}eg)^i_j\big)(g^{-1}\partial g)^j_i \\ &= \sum_{i,j=1}^{d} (g^{-1}\xi g)^i_j(g^{-1}\partial g)^j_i \\ &= \operatorname{tr}(g^{-1}\xi g g^{-1}\partial g) = \operatorname{tr}(\xi\partial) = \partial_\xi.\end{aligned}$$

□

Proposition 3.2.3. *The following diagram commutes.*

$$\begin{array}{ccc}
Ugl(d,\mathbb{C}) & \xleftarrow{\operatorname{Ad} g} & Ugl(d,\mathbb{C}) \\
{\scriptstyle \partial_\xi}\uparrow & & \uparrow{\scriptstyle \partial_{\xi\circ\operatorname{Ad} g}} \\
Ugl(d,\mathbb{C}) & \xleftarrow{\operatorname{Ad} g} & Ugl(d,\mathbb{C}) \\
\uparrow & & \uparrow \\
gl(d,\mathbb{C}) & \xleftarrow{\operatorname{Ad} g} & gl(d,\mathbb{C}) \\
{\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle \xi\circ\operatorname{Ad} g} \\
\mathbb{C} & = & \mathbb{C}
\end{array}$$

Proof. The following diagram commutes since the linear mapping $g^{-1}\partial g$ is the quantum derivation of the universal enveloping algebra with respect to the generating matrix $(\operatorname{Ad} g \otimes \delta)e = g^{-1}eg$ by Proposition 3.2.1.

$$
\begin{array}{ccc}
U\,gl(d,\mathbb{C}) & \xleftarrow{\operatorname{Ad} g} & U\,gl(d,\mathbb{C}) \\
{\scriptstyle (g^{-1}\partial g)_\xi}\uparrow & & \uparrow{\scriptstyle \partial_{\xi\circ\operatorname{Ad} g}} \\
U\,gl(d,\mathbb{C}) & \xleftarrow{\operatorname{Ad} g} & U\,gl(d,\mathbb{C}) \\
\uparrow & & \uparrow \\
gl(d,\mathbb{C}) & \xleftarrow{\operatorname{Ad} g} & gl(d,\mathbb{C}) \\
{\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle \xi\circ\operatorname{Ad} g} \\
\mathbb{C} & = & \mathbb{C}
\end{array}
$$

We have $\partial_\xi = (g^{-1}\partial g)_\xi$ by Proposition 3.2.2. □

Proposition 3.2.4. *We have* $\xi \circ \operatorname{Ad} g = g^{-1}\xi g$.

Proof. We have

$$
\begin{aligned}
(\xi \circ \operatorname{Ad} g)^i_j &= (\xi \circ \operatorname{Ad} g)(e^i_j) \\
&= \xi\big((\operatorname{Ad} g)e^i_j\big) \\
&= \xi\big((g^{-1}eg)^i_j\big) = (g^{-1}\xi g)^i_j.
\end{aligned}
$$

□

3.3 Proof of Main Theorem

We prove the main theorem in this section. We have

$$
[\partial^m_\xi x, \partial^n_\xi y] = \sum_{i_1,j_1=1}^{d} \cdots \sum_{i_{m+n},j_{m+n}=1}^{d} [\partial^{j_1}_{i_1} \cdots \partial^{j_m}_{i_m} x, \partial^{j_{m+1}}_{i_{m+1}} \cdots \partial^{j_{m+n}}_{i_{m+n}} y]\,\xi^{i_1}_{j_1} \cdots \xi^{i_{m+n}}_{j_{m+n}}
$$

and it is sufficient to show that we have $\left[\partial_\xi^m x, \partial_\xi^n y\right] = 0$ for any element ξ of the dense subset

$$\left\{ \xi \in gl(d,\mathbb{C})^* = M(d,\mathbb{C}) : \xi \text{ has } d \text{ distinct eigenvalues} \right\}$$

of the dual space $gl(d,\mathbb{C})^* = M(d,\mathbb{C})$. We may assume that we have

$$\xi = \operatorname{diag}(z_1,\dots,z_d), \qquad \prod_{i<j}(z_i - z_j) \neq 0$$

by Proposition 3.2.3 and 3.2.4. Vinberg [5] and Rybnikov [8] showed that the quantum argument shift algebra in the direction ξ coincides with the commutant of the set

$$\left\{ e_i^i, \sum_{j\neq i} \frac{e_i^j e_j^i}{z_i - z_j} \right\}_{i=1}^d .$$

We have the following.

Proposition 3.3.1. *It is sufficient to show the following for any i.*

1. *We have* $[\operatorname{ad} e_i^i, \partial_\xi] = 0$.
2. *We have* $(\operatorname{ad} \sum_{j\neq i} \frac{e_i^j e_j^i}{z_i - z_j})(\partial_\xi x) = 0$ *for any central element* x.
3. *We have* $[[\operatorname{ad} \sum_{j\neq i} \frac{e_i^j e_j^i}{z_i - z_j}, \partial_\xi], \partial_\xi] = 0$.

Proof. Suppose that x is an arbitrary central element of the universal enveloping algebra $Ugl(d,\mathbb{C})$. We have $(\operatorname{ad} e_i^i)(x) = 0$ since the element x is central. Then

$$0 = \partial_\xi^n (\operatorname{ad} e_i^i)(x) = (\operatorname{ad} e_i^i)(\partial_\xi^n x)$$

by the first condition $[\operatorname{ad} e_i^i, \partial_\xi] = 0$. Thus the element $\partial_\xi^n x$ commutes with the element e_i^i. Next the second condition implies

$$[\operatorname{ad} \sum_{j\neq i} \frac{e_i^j e_j^i}{z_i - z_j}, \partial_\xi](x) = 0$$

since the element x is central and the third condition implies

$$0 = \partial_\xi^{n-1}[\mathrm{ad} \sum_{j \neq i} \frac{e_i^j e_j^i}{z_i - z_j}, \partial_\xi](x) = [\mathrm{ad} \sum_{j \neq i} \frac{e_i^j e_j^i}{z_i - z_j}, \partial_\xi](\partial_\xi^{n-1} x). \tag{3.3}$$

We have $(\mathrm{ad} \sum_{j \neq i} \frac{e_i^j e_j^i}{z_i - z_j})(\partial_\xi x) = 0$ by the second condition. We proceed to show

$$(\mathrm{ad} \sum_{j \neq i} \frac{e_i^j e_j^i}{z_i - z_j})(\partial_\xi^n x) = 0$$

by induction on the positive integer n. Suppose that we have $n > 1$ and

$$(\mathrm{ad} \sum_{j \neq i} \frac{e_i^j e_j^i}{z_i - z_j})(\partial_\xi^{n-1} x) = 0.$$

We have

$$\begin{aligned} 0 &= \partial_\xi (\mathrm{ad} \sum_{j \neq i} \frac{e_i^j e_j^i}{z_i - z_j})(\partial_\xi^{n-1} x) \\ &= (\mathrm{ad} \sum_{j \neq i} \frac{e_i^j e_j^i}{z_i - z_j})(\partial_\xi^n x) - [\mathrm{ad} \sum_{j \neq i} \frac{e_i^j e_j^i}{z_i - z_j}, \partial_\xi](\partial_\xi^{n-1} x) \\ &= (\mathrm{ad} \sum_{j \neq i} \frac{e_i^j e_j^i}{z_i - z_j})(\partial_\xi^n x) \end{aligned}$$

by the equation (3.3). Therefore, the element $\partial_\xi^n x$ belongs to the quantum argument shift algebra in the direction ξ. □

Now we prove the main theorem.

Proof of Theorem 3.1.1. It is sufficient to show the conditions in Proposition 3.3.1.

1. We prove the first condition. Suppose that x is an arbitrary element of the

universal enveloping algebra $Ugl(d,\mathbb{C})$. We have

$$\begin{aligned}[\operatorname{ad} e^i_i, \partial_\xi](x) &= [e^i_i, \partial_\xi x] - \partial_\xi [e^i_i, x] \\ &= -[\partial_\xi e^i_i, x] - \operatorname{tr}\big(\xi[\partial e^i_i, \partial x]\big) \\ &= -\operatorname{tr}\big(\xi[E^i_i, \partial x]\big) \\ &= z_i(\partial x)^i_i - z_i(\partial x)^i_i = 0\end{aligned}$$

by the quantum Leibniz rule.

2. We prove the second condition. Suppose that x is an arbitrary central element of the universal enveloping algebra $Ugl(d,\mathbb{C})$. It is sufficient to show

$$(\operatorname{ad} \sum_{j\neq i} \frac{e^j_i e^i_j}{z_i - z_j}) \operatorname{tr}\big(\xi e^n\big) = 0$$

since the element $\partial_\xi x$ belongs to the C-module

$$\operatorname{span}_C\Big\{ \operatorname{tr}\big(\xi e^n\big) : n = 0, 1, 2, \dots \Big\}$$

by Corollary 2.5.1. We have

$$(\operatorname{ad} \sum_{j\neq i} \frac{e^j_i e^i_j}{z_i - z_j}) \operatorname{tr}(\xi e^n) = \sum_{j\neq i} \sum_{k=1}^{d} \frac{z_k}{z_i - z_j} \big[e^j_i e^i_j, (e^n)^k_k\big]$$

and

$$\begin{aligned}\sum_{k=1}^{d} z_k \big[e^j_i e^i_j, (e^n)^k_k\big] &= \sum_{k=1}^{d} z_k \Big(\big[e^j_i, (e^n)^k_k\big] e^i_j + e^j_i \big[e^i_j, (e^n)^k_k\big] \Big) \\ &= \sum_{k=1}^{d} z_k \Big(\big(\delta^j_k (e^n)^k_i - (e^n)^j_k \delta^k_i\big) e^i_j + e^j_i \big(\delta^i_k (e^n)^k_j - (e^n)^i_k \delta^k_j\big) \Big) \\ &= (z_i - z_j)\big(-(e^n)^j_i e^i_j + e^j_i (e^n)^i_j\big)\end{aligned}$$

by Proposition 2.4.1. We have

$$
\begin{aligned}
(\operatorname{ad}\sum_{j\neq i}\frac{e_i^j e_j^i}{z_i-z_j})\operatorname{tr}(\xi e^n) &= \sum_{j\neq i}\bigl(-(e^n)_i^j e_j^i + e_i^j (e^n)_j^i\bigr)\\
&= \sum_{j=1}^{d}\bigl(-(e^n)_i^j e_j^i + e_i^j (e^n)_j^i\bigr)\\
&= \sum_{j=1}^{d}\Bigl(-\bigl[(e^n)_i^j, e_j^i\bigr] + \bigl[e_i^j, (e^n)_j^i\bigr]\Bigr) - (e^{n+1})_i^i + (e^{n+1})_i^i\\
&= 0
\end{aligned}
$$

by Proposition 2.4.1.

3. We prove the third condition. We have

$$
\begin{aligned}
\partial_\xi \sum_{j\neq i}\frac{e_i^j e_j^i}{z_i-z_j} &= \operatorname{tr}\Bigl(\xi\sum_{j\neq i}\frac{\partial\bigl(e_i^j e_j^i\bigr)}{z_i-z_j}\Bigr)\\
&= \operatorname{tr}\Bigl(\xi\sum_{j\neq i}\frac{\bigl(\partial e_i^j\bigr)e_j^i + e_i^j\bigl(\partial e_j^i\bigr) + \bigl(\partial e_i^j\bigr)\bigl(\partial e_j^i\bigr)}{z_i-z_j}\Bigr)\\
&= \operatorname{tr}\Bigl(\xi\sum_{j\neq i}\frac{E_i^j e_j^i + e_i^j E_j^i + E_i^j E_j^i}{z_i-z_j}\Bigr)\\
&= \sum_{j\neq i}\frac{\operatorname{tr}\bigl(\xi E_i^j\bigr)e_j^i + e_i^j\operatorname{tr}\bigl(\xi E_j^i\bigr) + \operatorname{tr}\bigl(\xi E_i^i\bigr)}{z_i-z_j}\\
&= \sum_{j\neq i}\frac{\operatorname{tr}\bigl(z_i E_i^j\bigr)e_j^i + e_i^j\operatorname{tr}\bigl(z_j E_j^i\bigr) + \operatorname{tr}\bigl(z_i E_i^i\bigr)}{z_i-z_j}\\
&= \sum_{j\neq i}\frac{z_i}{z_i-z_j}. \qquad (3.4)
\end{aligned}
$$

Suppose that x is an arbitrary element of the universal enveloping algebra

$Ugl(d,\mathbb{C})$. We have

$$\begin{aligned}[\mathrm{ad}\sum_{j\neq i}\frac{e_i^j e_j^i}{z_i-z_j},\partial_\xi](x) &= [\sum_{j\neq i}\frac{e_i^j e_j^i}{z_i-z_j},\partial_\xi x]-\partial_\xi[\sum_{j\neq i}\frac{e_i^j e_j^i}{z_i-z_j},x] \\ &= -[\partial_\xi\sum_{j\neq i}\frac{e_i^j e_j^i}{z_i-z_j},x]-\operatorname{tr}\big(\xi[\partial\sum_{j\neq i}\frac{e_i^j e_j^i}{z_i-z_j},\partial x]\big) \\ &= -\operatorname{tr}\big(\xi[\partial\sum_{j\neq i}\frac{e_i^j e_j^i}{z_i-z_j},\partial x]\big)\end{aligned}$$

and

$$\begin{aligned}&[[\mathrm{ad}\sum_{j\neq i}\frac{e_i^j e_j^i}{z_i-z_j},\partial_\xi],\partial_\xi](x) \\ &= -\operatorname{tr}\big(\xi[\partial\sum_{j\neq i}\frac{e_i^j e_j^i}{z_i-z_j},\partial\partial_\xi x]\big)+\partial_\xi\operatorname{tr}\big(\xi[\partial\sum_{j\neq i}\frac{e_i^j e_j^i}{z_i-z_j},\partial x]\big) \\ &= \operatorname{tr}\big(\xi[\partial\partial_\xi\sum_{j\neq i}\frac{e_i^j e_j^i}{z_i-z_j},\partial x]\big) \\ &\qquad +\sum_{i_1,j_1,i_2,j_2=1}^{d}(z_{i_1}z_{i_2}-z_{j_1}z_{j_2})(\partial_{j_1}^{i_1}\partial_{j_2}^{i_2}\sum_{j\neq i}\frac{e_i^j e_j^i}{z_i-z_j})(\partial_{i_1}^{j_1}\partial_{i_2}^{j_2}x) \\ &= \sum_{i_1,j_1,i_2,j_2=1}^{d}(z_{i_1}z_{i_2}-z_{j_1}z_{j_2})(\partial_{j_1}^{i_1}\partial_{j_2}^{i_2}\sum_{j\neq i}\frac{e_i^j e_j^i}{z_i-z_j})(\partial_{i_1}^{j_1}\partial_{i_2}^{j_2}x)\end{aligned}$$

by the quantum Leibniz rule and the equation (3.4). We have

$$\begin{aligned}(z_{i_1}z_{i_2}-z_{j_1}z_{j_2})\partial_{j_1}^{i_1}\partial_{j_2}^{i_2}(e_i^j e_j^i) &= (z_{i_1}z_{i_2}-z_{j_1}z_{j_2})\big((\partial_{j_1}^{i_1}e_i^j)(\partial_{j_2}^{i_2}e_j^i)+(\partial_{j_2}^{i_2}e_i^j)(\partial_{j_1}^{i_1}e_j^i)\big) \\ &= 0.\end{aligned}$$ □

3.4 Quantum Argument Shift Algebras

Suppose that ξ is an arbitrary element of the dual space $gl(d,\mathbb{C})^*$. Recall that in Chapter 2 we defined C_ξ as the algebra generated by the set $\bigcup_{n=0}^{\infty}\partial_\xi^n C$. The algebra C_ξ is commutative by Theorem 3.1.1. We write $\bar{C}$ for the Lie–Poisson center of the symmetric algebra $Sgl(d,\mathbb{C})$ and $\bar{C}_\xi$ for the argument shift algebra in the direction ξ.

Theorem 3.4.1 ([18])**.** *The algebra C_ξ is the quantum argument shift algebra in the direction ξ.*

Proof. Both $\bar{C}$ and C are freely generated by $\big(\operatorname{tr} e, \dots, \operatorname{tr} e^d\big)$.

$$\bar{C} = \mathbb{C}\Big[\operatorname{tr} e, \dots, \operatorname{tr} e^d\Big] \simeq \mathbb{C}\big[x_1, \dots, x_d\big], \;\; C = \mathbb{C}\Big[\operatorname{tr} e, \dots, \operatorname{tr} e^d\Big] \simeq \mathbb{C}\big[x_1, \dots, x_d\big].$$

Here, the matrix e is considered as an element of $M\big(d, Sgl(d,\mathbb{C})\big)$ in the left-hand expression, and as an element of $M\big(d, Ugl(d,\mathbb{C})\big)$ in the right-hand expression. The additional term $\big((\partial x)(\partial y)\big)^i_j$ from the quantum Leibniz rule does not contribute to the highest-order term of $\partial^i_j(xy)$. Therefore, $\operatorname{gr} C_\xi = \bar{C}_\xi$. □

Chapter 4

Second-Order Quantum Argument Shifts

4.1 Introduction

Suppose that ξ is an arbitrary numerical matrix. Recall that

$$C_\xi = \lim_{n\to\infty} C_\xi^{(n)} = \bigcup_{n=0}^{\infty} C_\xi^{(n)}, \qquad C_\xi^{(0)} = C, \qquad C_\xi^{(n)} = C_\xi^{(n-1)}\big[\partial_\xi^n C\big]$$

is the quantum argument shift algebra in the direction ξ. We identified in Section 2.5 the generators of the algebra $C_\xi^{(1)}$:

$$C = \mathbb{C}\big[\operatorname{tr} e^n : n = 0, 1, 2, \ldots\big], \qquad C_\xi^{(1)} = C\Big[\operatorname{tr}\big(\xi e^n\big) : n = 1, 2, \ldots\Big].$$

We proceed to identify the generators of the algebra $C_\xi^{(2)}$. We will write ∂ for the algebraic homomorphism $\delta + \partial$ in Chapter 2 and Chapter 3 since it is more convenient to work with an algebraic homomorphism than with a linear mapping: the quantum derivation ∂ is the unique homomorphism of unital complex algebras of the universal enveloping algebra $U gl(d, \mathbb{C})$ into the matrix algebra

$M\bigl(d, Ugl(d,\mathbb{C})\bigr)$ such that $\partial\operatorname{tr}(\xi e) = \operatorname{tr}(\xi e) + \xi$ for any numerical matrix ξ. Theorem 2.4.1 and 2.5.1 and Corollary 2.5.1 should be modified as follows:

Theorem 4.1.1. *The quantum derivations of the matrix elements $(e^n)^i_j$ are given by the following formulae*

$$\begin{aligned}\partial(e^n)^i_j &= \sum_{m=0}^{n}\Bigl(f_+^{(n-m-1)}(e)_j(e^m)^i + f_-^{(n-m-1)}(e)(e^m)^i_j\Bigr)\\ &= \sum_{m=0}^{n}\Bigl((e^m)_j f_+^{(n-m-1)}(e)^i + (e^m)^i_j f_-^{(n-m-1)}(e)\Bigr)\end{aligned}$$

for any nonnegative integer n.

Theorem 4.1.2.

$$\partial\prod_m \operatorname{tr} e^{n_m} = \sum_{m_1=-1}^{n_1}\operatorname{tr} e^{m_1}\sum_{m_2=-1}^{n_2}\operatorname{tr} e^{m_2}\cdots\prod_k f_-^{(n_k-m_k-1)}(e)$$

for a finite product $\prod_m \operatorname{tr} e^{n_m}$.

Corollary 4.1.1.

$$\partial_\xi\prod_m \operatorname{tr} e^{n_m} = \sum_{m_1=-1}^{n_1}\operatorname{tr} e^{m_1}\sum_{m_2=-1}^{n_2}\operatorname{tr} e^{m_2}\cdots\operatorname{tr}\bigl(\xi\prod_k f_-^{(n_k-m_k-1)}(e)\bigr)$$

for a finite product $\prod_m \operatorname{tr} e^{n_m}$.

4.2 Formulae for Second-Order Quantum Argument Shifts

We present formulae for the second-order quantum argument shifts of central elements. Theorem 4.1.1 is still enough for this purpose. We adopt the convention $\operatorname{tr} e^{-1} = 1$ for simplicity of notation. The following formulae give the quantum argument shifts of an arbitrary central element up to the second order.

Theorem 4.2.1 ([19]).

$$\partial\partial_\xi \prod_m \operatorname{tr} e^{n_m} = \sum_{m_1=-1}^{n_1} \operatorname{tr} e^{m_1} \sum_{m_2=-1}^{n_2} \operatorname{tr} e^{m_2} \cdots \sum_{k_1=-1}^{n_1-m_1-1} f_-^{(k_1)}(e) \sum_{k_2=-1}^{n_2-m_2-1} f_-^{(k_2)}(e) \cdots \partial \operatorname{tr}\bigl(\xi \prod_\ell f_-^{(n_\ell-m_\ell-k_\ell-2)}(e)\bigr) \tag{4.1}$$

for a finite product $\prod_m \operatorname{tr} e^{n_m}$.

Proof. The proof is by direct computation:

$$\begin{aligned}
&\partial\partial_\xi\bigl(\operatorname{tr} e^{n_1} \operatorname{tr} e^{n_2} \cdots\bigr) \\
&= \sum_{k_1=-1}^{n_1} \sum_{k_2=-1}^{n_2} \cdots \partial\bigl(\prod_\ell \operatorname{tr} e^{k_\ell}\bigr)\partial\Bigl(\operatorname{tr}\bigl(\xi \prod_\ell f_-^{(n_\ell-k_\ell-1)}(e)\bigr)\Bigr) \\
&= \sum_{k_1=-1}^{n_1} \sum_{k_2=-1}^{n_2} \cdots \sum_{m_1=-1}^{k_1} \operatorname{tr} e^{m_1} \sum_{m_2=-1}^{k_2} \operatorname{tr} e^{m_2} \cdots \\
&\qquad \bigl(\prod_\ell f_-^{(k_\ell-m_\ell-1)}(e)\bigr)\partial \operatorname{tr}\bigl(\xi \prod_\ell f_-^{(n_\ell-k_\ell-1)}(e)\bigr)
\end{aligned}$$

by Corollary 4.1.1. Since

$$\sum_{k_1=-1}^{n_1} \sum_{k_2=-1}^{n_2} \cdots \sum_{m_1=-1}^{k_1} \sum_{m_2=-1}^{k_2} \cdots = \sum_{m_1=-1}^{n_1} \sum_{m_2=-1}^{n_2} \cdots \sum_{k_1=m_1}^{n_1} \sum_{k_2=m_2}^{n_2} \cdots,$$

we obtain the formula (4.1). □

We have

$$\partial_\xi^2 \prod_m \operatorname{tr} e^{n_m} = \sum_{m_1=-1}^{n_1} \operatorname{tr} e^{m_1} \sum_{m_2=-1}^{n_2} \operatorname{tr} e^{m_2} \cdots \sum_{k_1=-1}^{n_1-m_1-1} \sum_{k_2=-1}^{n_2-m_2-1} \cdots \operatorname{tr}\Bigl(\xi \prod_\ell f_-^{(k_\ell)}(e) \partial \operatorname{tr}\bigl(\xi \prod_\ell f_-^{(n_\ell-m_\ell-k_\ell-2)}(e)\bigr)\Bigr) \tag{4.2}$$

by the formula (4.1). The formula (4.2) implies Corollary 4.2.1:

Corollary 4.2.1 ([19]). *The algebra $C_\xi^{(2)}$ is contained in the algebra generated by the algebra $C_\xi^{(1)}$ and the elements*

$$\operatorname{tr}\Big(\xi e^m \partial \operatorname{tr}\big(\xi e^n\big)\Big) + \operatorname{tr}\Big(\xi e^n \partial \operatorname{tr}\big(\xi e^m\big)\Big), \qquad m, n = 0, 1, 2, \ldots.$$

Proof. The elements of the form

$$\sum_{m_1=-1}^{n_1+1} \sum_{m_2=-1}^{n_2+1} \cdots \operatorname{tr}\Big(\xi \prod_k f_-^{(m_k)}(e) \partial \operatorname{tr}\big(\xi \prod_k f_-^{(n_k-m_k)}(e)\big)\Big)$$

belong to the additive monoid generated by the elements

$$\operatorname{tr}\Big(\xi e^n \partial \operatorname{tr}\big(\xi e^n\big)\Big), \ \operatorname{tr}\Big(\xi e^m \partial \operatorname{tr}\big(\xi e^n\big)\Big) + \operatorname{tr}\Big(\xi e^n \partial \operatorname{tr}\big(\xi e^m\big)\Big), \ m, n = 0, 1, 2, \ldots.$$

Any element of the algebra $C_\xi^{(2)}$ is contained in the algebra generated by the algebra $C_\xi^{(1)}$ and the elements

$$\operatorname{tr}\Big(\xi e^m \partial \operatorname{tr}\big(\xi e^n\big)\Big) + \operatorname{tr}\Big(\xi e^n \partial \operatorname{tr}\big(\xi e^m\big)\Big), \qquad m, n = 0, 1, 2, \ldots.$$

□

Suppose that m and n are nonnegative integers. We have

$$\begin{aligned}\operatorname{tr}\Big(\xi e^m \partial \operatorname{tr}\big(\xi e^n\big)\Big) &= \operatorname{tr}\Bigg(\xi e^m \sum_{j=1}^{n+1}\Big(f_+^{(n-j)}(e)\xi e^{j-1} + f_-^{(n-j)}(e)\operatorname{tr}\big(\xi e^{j-1}\big)\Big)\Bigg) \\ &= \sum_{j=1}^{n+1}\Big(\operatorname{tr}\big(\xi e^m f_+^{(n-j)}(e)\xi e^{j-1}\big) + \operatorname{tr}\big(\xi e^m f_-^{(n-j)}(e)\big)\operatorname{tr}\big(\xi e^{j-1}\big)\Big)\end{aligned}$$

by Theorem 4.1.1 and thus

$$\operatorname{tr}\Big(\xi e^m \partial \operatorname{tr}\big(\xi e^n\big)\Big) = \sum_{j=1}^{n} \operatorname{tr}\big(\xi e^m f_+^{(n-j)}(e)\xi e^{j-1}\big) \mod C_\xi^{(1)} \tag{4.3}$$

by Theorem 2.5.3.

Definition 4.2.1. We define the $m+n$ by n integer matrix $P_n^{(m)}$ as the coefficients of the polynomials: $x^m f_+^{(n-j)}(x) = \sum_{i=1}^{m+n} \Big(P_n^{(m)}\Big)_j^i x^{i-1}$ and let $P_n = P_n^{(0)}$.

The matrix P_n is the submatrix of the matrix P_{n+1} taking the top right corner: $P_{n+1} = \begin{pmatrix} * & P_n \\ 1 & 0 \end{pmatrix}$ and $P_n^{(m)} = \begin{pmatrix} 0 \\ P_n \end{pmatrix}$ (the first m row vectors are null). Since for instance

$$\begin{pmatrix} f_+^{(3)}(x) & f_+^{(2)}(x) & f_+^{(1)}(x) & f_+^{(0)}(x) \end{pmatrix} = \begin{pmatrix} 3x + x^3 & 1 + x^2 & x & 1 \end{pmatrix}$$
$$= \begin{pmatrix} x^0 & x^1 & x^2 & x^3 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

we have $P_4 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

Definition 4.2.2. We define

$$\tau_\xi(x) = \operatorname{tr}\left(\begin{pmatrix} \xi & \xi e & \cdots & \xi e^{m-1} \end{pmatrix} x \begin{pmatrix} \xi \\ \xi e \\ \vdots \\ \xi e^{n-1} \end{pmatrix} \right) = \sum_{i=1}^{m} \sum_{j=1}^{n} x_j^i \operatorname{tr}\big(\xi e^{i-1} \xi e^{j-1}\big)$$

for any m by n numerical matrix x.

Now by the formula (4.3) we have

$$\operatorname{tr}\Big(\xi e^m \partial \operatorname{tr}\big(\xi e^n\big)\Big) = \tau_\xi\Big(P_n^{(m)}\Big) \mod C_\xi^{(1)}. \tag{4.4}$$

4.3 Generators of the algebra $C_\xi^{(2)}$

We give the reduced set of generators of the algebra $C_\xi^{(2)}$. The generators given in Corollary 4.2.1 can be expressed with the help of lower triangular matrices.

Definition 4.3.1. Suppose that n is a nonnegative integer and x is an n by n numerical matrix. We define the n by n lower triangular numerical matrix $\sigma(x)$ by the formula

$$\sigma(x) = \begin{pmatrix} x_1^1 & 0 & \cdots & 0 \\ x_1^2 + x_2^1 & x_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^n + x_n^1 & x_2^n + x_n^2 & \cdots & x_n^n \end{pmatrix} = \sum_{i,j=1}^{n} x_j^i \delta_{\max\{i,j\}} \delta^{\min\{i,j\}}.$$

Proposition 4.3.1. $(\tau_\xi \circ \sigma)(x) = \tau_\xi(x)$ *for any square numerical matrix* x.

Proof. We have

$$\begin{aligned}(\tau_\xi \circ \sigma)(x) &= \sum_{i,j=1}^{n} x_j^i \operatorname{tr}\left(\xi e^{\max\{i,j\}-1} \xi e^{\min\{i,j\}-1}\right) \\ &= \sum_{i,j=1}^{n} x_j^i \operatorname{tr}\left(\xi e^{i-1} \xi e^{j-1}\right) = \tau_\xi(x)\end{aligned}$$

for any n by n numerical matrix x by Proposition 2.5.1. □

Proposition 4.3.2. *For any nonnegative integers* m *and* n *we have*

$$\operatorname{tr}\Big(\xi e^m \partial \operatorname{tr}\big(\xi e^n\big)\Big) + \operatorname{tr}\Big(\xi e^n \partial \operatorname{tr}\big(\xi e^m\big)\Big) = (\tau_\xi \circ \sigma) \begin{pmatrix} 0 & P_n^T \\ P_m & 0 \end{pmatrix} \mod C_\xi^{(1)}.$$

Proof. We have

$$
\begin{aligned}
\operatorname{tr}\Big(\xi e^m \partial \operatorname{tr}\big(\xi e^n\big)\Big) + \operatorname{tr}\Big(\xi e^n \partial \operatorname{tr}\big(\xi e^m\big)\Big) &= \tau_\xi\Big(P_n^{(m)}\Big) + \tau_\xi\Big(P_m^{(n)}\Big) \\
&= \tau_\xi\left(\begin{array}{cc} & \begin{array}{cc} n & m \end{array} \\ \begin{array}{c} m \\ n \end{array} & \begin{pmatrix} 0 & 0 \\ P_n & 0 \end{pmatrix} \end{array} \right) + \tau_\xi\left(\begin{array}{cc} & \begin{array}{cc} m & n \end{array} \\ \begin{array}{c} n \\ m \end{array} & \begin{pmatrix} 0 & 0 \\ P_m & 0 \end{pmatrix} \end{array} \right) \\
&= (\tau_\xi \circ \sigma)\left(\begin{array}{cc} & \begin{array}{cc} m & n \end{array} \\ \begin{array}{c} n \\ m \end{array} & \begin{pmatrix} 0 & P_n^T \\ 0 & 0 \end{pmatrix} \end{array} \right) + (\tau_\xi \circ \sigma)\left(\begin{array}{cc} & \begin{array}{cc} m & n \end{array} \\ \begin{array}{c} n \\ m \end{array} & \begin{pmatrix} 0 & 0 \\ P_m & 0 \end{pmatrix} \end{array} \right) \\
&= (\tau_\xi \circ \sigma)\left(\begin{array}{cc} & \begin{array}{cc} m & n \end{array} \\ \begin{array}{c} n \\ m \end{array} & \begin{pmatrix} 0 & P_n^T \\ P_m & 0 \end{pmatrix} \end{array} \right) \mod C_\xi^{(1)}
\end{aligned}
$$

by the formula (4.4) and Proposition 4.3.1. □

The following theorem plays an essential role in reducing the number of the generators given in Corollary 4.2.1 and Proposition 4.3.2. The proof will appear in Section 4.4.

Theorem 4.3.1 ([19])**.** *For any nonnegative integers m and n we have*

$$
\sigma\begin{pmatrix} 0 & P_m^T \\ P_{m+2n} & 0 \end{pmatrix} = \sum_{k=0}^{n}\left(\binom{2n-k}{k} + \binom{2n-k-1}{k-1}\right)P_{m+k}^{(m+k)} \tag{4.5}
$$

and

$$
\sigma\begin{pmatrix} 0 & P_m^T \\ P_{m+2n+1} & 0 \end{pmatrix} = \sum_{k=0}^{n}\binom{2n-k}{k}\Big(P_{m+k+1}^{(m+k)} + P_{m+k}^{(m+k+1)}\Big). \tag{4.6}
$$

We apply the theorem to Proposition 4.3.2 and obtain the following.

Corollary 4.3.1 ([19]). *For any nonnegative integers m and n we have*

$$\operatorname{tr}\Big(\xi e^{m+2n}\partial\operatorname{tr}\big(\xi e^{m}\big)\Big)+\operatorname{tr}\Big(\xi e^{m}\partial\operatorname{tr}\big(\xi e^{m+2n}\big)\Big) = \sum_{k=0}^{n}\left(\binom{2n-k}{k}+\binom{2n-k-1}{k-1}\right)\tau_\xi\Big(P_{m+k}^{(m+k)}\Big) \mod C_\xi^{(1)}$$

and

$$\operatorname{tr}\Big(\xi e^{m+2n+1}\partial\operatorname{tr}\big(\xi e^{m}\big)\Big)+\operatorname{tr}\Big(\xi e^{m}\partial\operatorname{tr}\big(\xi e^{m+2n+1}\big)\Big) = \sum_{k=0}^{n}\binom{2n-k}{k}\tau_\xi\Big(P_{m+k+1}^{(m+k)}+P_{m+k}^{(m+k+1)}\Big) \mod C_\xi^{(1)}.$$

We see that the sets

$$\Big\{\sigma\big(P_n\big)\Big\}_{n=1}^{\infty}, \qquad \Big\{P_n^{(n-1)}+P_{n-1}^{(n)},P_n^{(n)}\Big\}_{n=1}^{\infty}$$

are bases of the vector space $\operatorname{span}\Big\{P_m^{(n)}+P_n^{(m)}\Big\}_{m,n=0}^{\infty}$.

1. We have

$$\sigma\big(P_{2n-1}\big)=\sum_{m=1}^{n}\binom{2n-m-1}{m-1}\Big(P_m^{(m-1)}+P_{m-1}^{(m)}\Big)$$

by Theorem 4.3.1 and

$$\left(\begin{pmatrix}1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & 1\\ 2 & 0\end{pmatrix}, \begin{pmatrix}1 & 0 & 0\\ 0 & 3 & 0\\ 7 & 0 & 1\\ 0 & 2 & 0\\ 2 & 0 & 0\end{pmatrix}, \dots\right) = \Big(\sigma\big(P_1\big), \sigma\big(P_3\big), \sigma\big(P_5\big), \dots\Big)$$

$$= \Big(P_1, P_2^{(1)} + P_1^{(2)}, P_3^{(2)} + P_2^{(3)}, \dots\Big)\left(\binom{2n-m-1}{m-1}\right)_{1\le m\le n}$$

$$= \left(\begin{pmatrix}1\end{pmatrix}, \begin{pmatrix}0 & 0\\ 0 & 1\\ 2 & 0\end{pmatrix}, \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 1 & 0 & 1\\ 0 & 2 & 0\\ 2 & 0 & 0\end{pmatrix}, \dots\right)\begin{pmatrix}1 & 1 & 1 & \cdots\\ 0 & 1 & 3 & \cdots\\ 0 & 0 & 1 & \cdots\\ \vdots & \vdots & \vdots & \end{pmatrix}.$$

We have

$$\begin{pmatrix}1 & 0 & 0\\ 0 & 3 & 0\\ 7 & 0 & 1\\ 0 & 2 & 0\\ 2 & 0 & 0\end{pmatrix} = 1\begin{pmatrix}1\end{pmatrix} + 3\begin{pmatrix}0 & 0\\ 0 & 1\\ 2 & 0\end{pmatrix} + 1\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 1 & 0 & 1\\ 0 & 2 & 0\\ 2 & 0 & 0\end{pmatrix}$$

for instance. We have

$$\left(\begin{pmatrix} 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \dots \right) = \left(P_1, P_2^{(1)} + P_1^{(2)}, P_3^{(2)} + P_2^{(3)}, \dots \right)$$

$$= \left(\sigma\left(P_1\right), \sigma\left(P_3\right), \sigma\left(P_5\right), \dots \right) \begin{pmatrix} 1 & 1 & 1 & \cdots \\ 0 & 1 & 3 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}^{-1}$$

$$= \left(\begin{pmatrix} 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 7 & 0 & 1 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \dots \right) \begin{pmatrix} 1 & -1 & 2 & \cdots \\ 0 & 1 & -3 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix} = 2 \begin{pmatrix} 1 \end{pmatrix} - 3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 0 \end{pmatrix} + 1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 7 & 0 & 1 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

for instance.

2. We have

$$\sigma\left(P_{2n}\right) = \sum_{m=1}^{n} \left(\binom{2n-m}{m} + \binom{2n-m-1}{m-1} \right) P_m^{(m)}$$

by Theorem 4.3.1 and

$$\left(\begin{pmatrix}0\\2\end{pmatrix}, \begin{pmatrix}0&0\\4&0\\0&2\\2&0\end{pmatrix}, \begin{pmatrix}0&0&0\\6&0&0\\0&9&0\\11&0&2\\0&2&0\\2&0&0\end{pmatrix}, \dots\right) = \Big(\sigma(P_2), \sigma(P_4), \sigma(P_6), \dots\Big)$$

$$= \left(P_1^{(1)}, P_2^{(2)}, P_3^{(3)}, \dots\right)\left(\binom{2n-m}{m} + \binom{2n-m-1}{m-1}\right)_{1\le m\le n}$$

$$= \left(\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}0&0\\0&0\\0&1\\1&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\\1&0&1\\0&1&0\\1&0&0\end{pmatrix}, \dots\right)\begin{pmatrix}2&4&6&\cdots\\0&2&9&\cdots\\0&0&2&\cdots\\\vdots&\vdots&\vdots&\end{pmatrix}.$$

We have

$$\begin{pmatrix}0&0&0\\6&0&0\\0&9&0\\11&0&2\\0&2&0\\2&0&0\end{pmatrix} = 6\begin{pmatrix}0\\1\end{pmatrix} + 9\begin{pmatrix}0&0\\0&0\\0&1\\1&0\end{pmatrix} + 2\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\\1&0&1\\0&1&0\\1&0&0\end{pmatrix}$$

for instance. We have

$$\left(\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}0&0\\0&0\\0&1\\1&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\\1&0&1\\0&1&0\\1&0&0\end{pmatrix}, \dots\right) = \left(P_1^{(1)}, P_2^{(2)}, P_3^{(3)}, \dots\right)$$

$$= \left(\sigma\left(P_2\right), \sigma\left(P_4\right), \sigma\left(P_6\right), \dots\right) \begin{pmatrix}2&4&6&\cdots\\0&2&9&\cdots\\0&0&2&\cdots\\\vdots&\vdots&\vdots&\end{pmatrix}^{-1}$$

$$= \left(\begin{pmatrix}0\\2\end{pmatrix}, \begin{pmatrix}0&0\\4&0\\0&2\\2&0\end{pmatrix}, \begin{pmatrix}0&0&0\\6&0&0\\0&9&0\\11&0&2\\0&2&0\\2&0&0\end{pmatrix}, \dots\right) \begin{pmatrix}\frac{1}{2}&-1&3&\cdots\\0&\frac{1}{2}&-\frac{9}{4}&\cdots\\0&0&\frac{1}{2}&\cdots\\\vdots&\vdots&\vdots&\end{pmatrix}$$

and

$$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\\1&0&1\\0&1&0\\1&0&0\end{pmatrix} = 3\begin{pmatrix}0\\2\end{pmatrix} - \frac{9}{4}\begin{pmatrix}0&0\\4&0\\0&2\\2&0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0&0&0\\6&0&0\\0&9&0\\11&0&2\\0&2&0\\2&0&0\end{pmatrix}$$

for instance.

The following theorem is the main result of this chapter.

Theorem 4.3.2 ([19])**.**

$$C_\xi^{(2)} = C_\xi^{(1)}\Big[\tau_\xi\Big(P_n^{(n)}\Big), \tau_\xi\Big(P_{n+1}^{(n)}\Big) + \tau_\xi\Big(P_n^{(n+1)}\Big) : n = 1, 2, \dots\Big].$$

Proof. The algebra $C_\xi^{(2)}$ is contained in the algebra

$$C_\xi^{(1)}\Big[\tau_\xi\Big(P_n^{(n)}\Big), \tau_\xi\Big(P_{n+1}^{(n)}\Big) + \tau_\xi\Big(P_n^{(n+1)}\Big) : n = 1, 2, \dots\Big]$$

by Corollary 4.2.1 and 4.3.1. We have

$$C_\xi^{(1)}\Big[\tau_\xi\Big(P_n^{(n)}\Big), \tau_\xi\Big(P_n^{(n-1)} + P_{n-1}^{(n)}\Big) : n = 1, 2, \dots\Big] = C_\xi^{(1)}\Big[\tau_\xi\big(P_n\big) : n = 1, 2, \dots\Big]$$

by the observation above. We prove that the elements $\tau_\xi\big(P_n\big)$ belong to the algebra

$$C_\xi^{(1)}\Big[\partial_\xi^2 \operatorname{tr} e^n : n = 3, 4, \dots\Big] \tag{4.7}$$

by induction on the nonnegative integer n. Suppose that the integer n is positive and the element $\tau_\xi\big(P_m\big)$ belongs to the algebra (4.7) for any nonnegative integer $m < n$. The element $\tau_\xi\big(P_n\big)$ belongs to the algebra (4.7) since the element

$$\begin{aligned}
&\partial_\xi^2 \operatorname{tr} e^{n+1} - (n+1)\tau_\xi\big(P_n\big) \\
&\qquad = \sum_{m=-1}^{n+1} \operatorname{tr} e^m \sum_{k=-1}^{n-m} \operatorname{tr}\Big(\xi f_-^{(k)}(e)\partial \operatorname{tr}\big(\xi f_-^{(n-m-k-1)}(e)\big)\Big) - (n+1)\tau_\xi\big(P_n\big)
\end{aligned}$$

belongs to the algebra $C_\xi^{(1)}\Big[\tau_\xi\big(P_m\big) : m < n\Big]$. □

We compute the first several elements of the generators:

$$\begin{aligned}
\tau_\xi\Big(P_1^{(1)}\Big) &= \mathrm{tr}\big(\xi^2 e\big),\\
\tau_\xi\Big(P_2^{(1)}\Big)+\tau_\xi\Big(P_1^{(2)}\Big) &= \mathrm{tr}\big(2\xi^2e^2+\xi e\xi e\big),\\
\tau_\xi\Big(P_2^{(2)}\Big) &= \mathrm{tr}\big(\xi^2e^3+\xi e\xi e^2\big),\\
\tau_\xi\Big(P_3^{(2)}\Big)+\tau_\xi\Big(P_2^{(3)}\Big) &= \mathrm{tr}\big(2\xi^2e^4+2\xi e\xi e^3+\xi e^2\xi e^2+\xi^2e^2\big),\\
\tau_\xi\Big(P_3^{(3)}\Big) &= \mathrm{tr}\big(\xi^2e^5+\xi e\xi e^4+\xi e^2\xi e^3+\xi^2e^3\big),\\
\tau_\xi\Big(P_4^{(3)}\Big)+\tau_\xi\Big(P_3^{(4)}\Big) &= \mathrm{tr}\big(2\xi^2e^6+2\xi e\xi e^5+2\xi e^2\xi e^4+\xi e^3\xi e^3+4\xi^2e^4+\xi e\xi e^3\big),\\
\tau_\xi\Big(P_4^{(4)}\Big) &= \mathrm{tr}\big(\xi^2e^7+\xi e\xi e^6+\xi e^2\xi e^5+\xi e^3\xi e^4+3\xi^2e^5+\xi e\xi e^4\big),\ldots.
\end{aligned}$$

They form a commutative family together with the elements $\Big\{\mathrm{tr}\big(\xi e^n\big)\ :\ n = 1,2,\ldots\Big\}$ (see Theorem 3.4.1 and 2.5.3).

4.4 Proof of Theorem 4.3.1

We note that the relations (4.5) and (4.6) for $m+1$ are equivalent to the same relations for m together with the relations on the first column vectors

$$\sigma\begin{pmatrix} 0 & P_{m+1}^T \\ P_{m+2n+1} & 0 \end{pmatrix}_1^i = \sum_{k=0}^{n}\left(\binom{2n-k}{k}+\binom{2n-k-1}{k-1}\right)\Big(P_{m+k+1}^{(m+k+1)}\Big)_1^i \tag{4.8}$$

and

$$\sigma\begin{pmatrix} 0 & P_{m+1}^T \\ P_{m+2n+2} & 0 \end{pmatrix}_1^i = \sum_{k=0}^{n}\binom{2n-k}{k}\Big(P_{m+k+2}^{(m+k+1)}+P_{m+k+1}^{(m+k+2)}\Big)_1^i. \tag{4.9}$$

The relations (4.8) and (4.9) are equivalent to the polynomial relations

$$
\begin{aligned}
f_+^{(m+2n)}(x) + f_+^{(m)}(x)x^{2n} &= \sum_{i=1}^{m+2n+1} \left(P_{m+2n+1} + P_{m+1}^{(2n)}\right)_1^i x^{i-1} \\
&= \sum_{k=0}^{n} \left(\binom{2n-k}{k} + \binom{2n-k-1}{k-1} \right) \sum_{i=1}^{m+2k+1} \left(P_{m+k+1}^{(k)}\right)_1^i x^{i-1} \\
&= \sum_{k=0}^{n} \left(\binom{2n-k}{k} + \binom{2n-k-1}{k-1} \right) f_+^{(m+k)}(x)x^k
\end{aligned}
$$

and

$$
\begin{aligned}
f_+^{(m+2n+1)}(x) + f_+^{(m)}(x)x^{2n+1} &= \sum_{i=1}^{m+2n+2} \left(P_{m+2n+2} + P_{m+1}^{(2n+1)}\right)_1^i x^{i-1} \\
&= \sum_{k=0}^{n} \binom{2n-k}{k} \sum_{i=1}^{m+2k+2} \left(P_{m+k+2}^{(k)} + P_{m+k+1}^{(k+1)}\right)_1^i x^{i-1} \\
&= \sum_{k=0}^{n} \binom{2n-k}{k} \left(f_+^{(m+k+1)}(x)x^k + f_+^{(m+k)}(x)x^{k+1} \right).
\end{aligned}
$$

We obtained the following.

Proposition 4.4.1. *Theorem 4.3.1 is equivalent to the following conditions.*

1. *For any nonnegative integer n we have*

$$
\sigma\left(P_{2n}\right) = \sum_{m=1}^{n} \left(\binom{2n-m}{m} + \binom{2n-m-1}{m-1} \right) P_m^{(m)} \tag{4.10}
$$

and

$$
\sigma\left(P_{2n+1}\right) = \sum_{m=0}^{n} \binom{2n-m}{m} \left(P_{m+1}^{(m)} + P_m^{(m+1)} \right). \tag{4.11}
$$

2. *For any nonnegative integers m and n we have*

$$
\begin{aligned}
&f_+^{(m+2n)}(x) + f_+^{(m)}(x)x^{2n} \\
&\quad = \sum_{k=0}^{n} \left(\binom{2n-k}{k} + \binom{2n-k-1}{k-1} \right) f_+^{(m+k)}(x)x^k
\end{aligned}
$$

and

$$\begin{aligned}f_+^{(m+2n+1)}(x) &+ f_+^{(m)}(x)x^{2n+1}\\ &= \sum_{k=0}^{n}\binom{2n-k}{k}\Big(f_+^{(m+k+1)}(x)x^k + f_+^{(m+k)}(x)x^{k+1}\Big).\end{aligned}$$

We write the relations (4.10) and (4.11) in terms of binomial coefficients.

Proposition 4.4.2. *The relation* (4.10) *is equivalent to the combinatorial relation*

$$\binom{2n_1+n_2+2n_3+1}{2n_3} + \binom{n_2+2n_3}{2n_3} = \sum_{n_4=0}^{n_3} \left(\binom{n_1+n_2+n_3+n_4+1}{2n_4} + \binom{n_1+n_2+n_3+n_4}{2n_4}\right)\binom{n_1+n_3-n_4}{2(n_3-n_4)}$$

comparing the $(2n_1+n_2+2, n_2+1)$ *elements of the relation* (4.10) *for*

$$n = n_1+n_2+n_3+1.$$

Proof. Suppose that n_1 and n_2 are nonnegative integers. The condition

$$(2n_1+n_2+2) + (n_2+1) \le 2n+1 \tag{4.12}$$

is necessary for the $(2n_1+n_2+2, n_2+1)$ element of the matrix $\sigma\big(P_{2n}\big)$ not to vanish. The condition (4.12) is equivalent to the condition

$$n_3 = n - n_1 - n_2 - 1 \ge 0$$

and the $(2n_1+n_2+2, n_2+1)$ element of the matrix $\sigma\big(P_{2n}\big)$ is given by the nonnegative integer

$$\begin{aligned}\binom{2n-(n_2+1)}{2n_1+n_2+1} &+ \binom{2n-(2n_1+n_2+2)}{n_2}\\ &= \binom{2n_1+n_2+2n_3+1}{2n_1+n_2+1} + \binom{n_2+2n_3}{n_2}\\ &= \binom{2n_1+n_2+2n_3+1}{2n_3} + \binom{n_2+2n_3}{2n_3}.\end{aligned}$$

The condition

$$(2n_1 + n_2 + 2) + (n_2 + 1) \leq 2m + 1 \tag{4.13}$$

is necessary for the $(2n_1+n_2+2, n_2+1)$ element of the matrix $P_m^{(m)}$ not to vanish. The condition (4.13) is equivalent to the condition

$$n_4 = n - m \leq n_3$$

and the $(2n_1 + n_2 + 2, n_2 + 1)$ element of the matrix

$$\sum_{m=1}^{n} \left(\binom{2n-m}{m} + \binom{2n-m-1}{m-1} \right) P_m^{(m)}$$

is given by the nonnegative integer

$$\begin{aligned}
&\sum_{n_4=0}^{n_3} \left(\binom{2n-(n-n_4)}{n-n_4} + \binom{2n-(n-n_4)-1}{n-n_4-1} \right) \binom{n-n_4-(n_2+1)}{2n_1+n_2+1-(n-n_4)} \\
&= \sum_{n_4=0}^{n_3} \left(\binom{n_1+n_2+n_3+n_4+1}{n_1+n_2+n_3-n_4+1} + \binom{n_1+n_2+n_3+n_4}{n_1+n_2+n_3-n_4} \right) \\
&\qquad \binom{n_1+n_3-n_4}{n_1-n_3+n_4} \\
&= \sum_{n_4=0}^{n_3} \left(\binom{n_1+n_2+n_3+n_4+1}{2n_4} + \binom{n_1+n_2+n_3+n_4}{2n_4} \right) \\
&\qquad \binom{n_1+n_3-n_4}{2(n_3-n_4)}.
\end{aligned}$$

□

Proposition 4.4.3. *The relation (4.11) is equivalent to the combinatorial relation*

$$\begin{aligned}
&\binom{2n_1+n_2+2n_3+2}{2n_3} + \binom{n_2+2n_3}{2n_3} = \sum_{n_4=0}^{n_3} \\
&\qquad \binom{n_1+n_2+n_3+n_4+1}{2n_4} \left(\binom{n_1+n_3-n_4+1}{2(n_3-n_4)} + \binom{n_1+n_3-n_4}{2(n_3-n_4)} \right)
\end{aligned}$$

comparing the $(2n_1 + n_2 + 3, n_2 + 1)$ elements of the relation (4.10) for

$$n = n_1 + n_2 + n_3 + 1.$$

Proof. Suppose that n_1 and n_2 are nonnegative integers. The condition

$$(2n_1 + n_2 + 3) + (n_2 + 1) \leq 2n + 2 \tag{4.14}$$

is necessary for the $(2n_1 + n_2 + 3, n_2 + 1)$ element of the matrix $\sigma\big(P_{2n+1}\big)$ not to vanish. The condition (4.14) is equivalent to the condition

$$n_3 = n - n_1 - n_2 - 1 \geq 0$$

and the $(2n_1 + n_2 + 3, n_2 + 1)$ element of the matrix $\sigma\big(P_{2n+1}\big)$ is given by the nonnegative integer

$$\begin{aligned} &\binom{2n + 1 - (n_2 + 1)}{2n_1 + n_2 + 2} + \binom{2n + 1 - (2n_1 + n_2 + 3)}{n_2} \\ &\quad = \binom{2n_1 + n_2 + 2n_3 + 2}{2n_1 + n_2 + 2} + \binom{n_2 + 2n_3}{n_2} \\ &\quad = \binom{2n_1 + n_2 + 2n_3 + 2}{2n_3} + \binom{n_2 + 2n_3}{2n_3}. \end{aligned}$$

The condition

$$(2n_1 + n_2 + 3) + (n_2 + 1) \leq 2m + 2 \tag{4.15}$$

is necessary for the $(2n_1 + n_2 + 3, n_2 + 1)$ element of the matrix $P^{(m)}_{m+1} + P^{(m+1)}_{m}$ not to vanish. The condition (4.15) is equivalent to the condition

$$n_4 = n - m \leq n_3$$

and the $(2n_1 + n_2 + 3, n_2 + 1)$ element of the matrix

$$\sum_{m=0}^{n} \binom{2n - m}{m} \Big(P^{(m)}_{m+1} + P^{(m+1)}_{m}\Big)$$

is given by the nonnegative integer

$$\sum_{n_4=0}^{n_3}\binom{2n-(n-n_4)}{n-n_4}$$
$$\left(\binom{n-n_4+1-(n_2+1)}{2n_1+n_2+2-(n-n_4)}+\binom{n-n_4-(n_2+1)}{2n_1+n_2+2-(n-n_4+1)}\right)$$
$$=\sum_{n_4=0}^{n_3}\binom{n_1+n_2+n_3+n_4+1}{n_1+n_2+n_3-n_4+1}$$
$$\left(\binom{n_1+n_3-n_4+1}{n_1-n_3+n_4+1}+\binom{n_1+n_3-n_4}{n_1-n_3+n_4}\right)$$
$$=\sum_{n_4=0}^{n_3}\binom{n_1+n_2+n_3+n_4+1}{2n_4}$$
$$\left(\binom{n_1+n_3-n_4+1}{2(n_3-n_4)}+\binom{n_1+n_3-n_4}{2(n_3-n_4)}\right).$$

□

We obtained the following.

Proposition 4.4.4. *Theorem 4.3.1 is equivalent to the following conditions.*

1. *For any nonnegative integers* $(n_k)_{k=1}^3$ *we have*

$$\binom{2n_1+n_2+2n_3+1}{2n_3}+\binom{n_2+2n_3}{2n_3}=\sum_{n_4=0}^{n_3}$$
$$\left(\binom{n_1+n_2+n_3+n_4+1}{2n_4}+\binom{n_1+n_2+n_3+n_4}{2n_4}\right)\binom{n_1+n_3-n_4}{2(n_3-n_4)}$$

and

$$\binom{2n_1+n_2+2n_3+2}{2n_3}+\binom{n_2+2n_3}{2n_3}=\sum_{n_4=0}^{n_3}$$
$$\binom{n_1+n_2+n_3+n_4+1}{2n_4}\left(\binom{n_1+n_3-n_4+1}{2(n_3-n_4)}+\binom{n_1+n_3-n_4}{2(n_3-n_4)}\right).$$

2. For any nonnegative integers m and n we have

$$f_+^{(m+2n)}(x) + f_+^{(m)}(x)x^{2n} = \sum_{k=0}^{n}\left(\binom{2n-k}{k} + \binom{2n-k-1}{k-1}\right) f_+^{(m+k)}(x)x^k$$

and

$$f_+^{(m+2n+1)}(x) + f_+^{(m)}(x)x^{2n+1} = \sum_{k=0}^{n}\binom{2n-k}{k}\left(f_+^{(m+k+1)}(x)x^k + f_+^{(m+k)}(x)x^{k+1}\right).$$

Proof of Theorem 4.3.1. We verify the corresponding conditions in Proposition 4.4.4 with Mathematica:

```
In[1]:= FullSimplify[Binomial[2n+m+2l+1,2l]+
  Binomial[m+2l,2l]-
  Sum[(Binomial[n+m+l+k+1,2k]+Binomial[n+m+l+k,2k])
  Binomial[n+l-k,2(l-k)],{k,0,l}],
  Element[n|m|l,Integers]&&n>=0&&m>=0&&l>=0]
Out[1]= 0
In[2]:= FullSimplify[Binomial[2n+m+2l+2,2l]+
  Binomial[m+2l,2l]-
  Sum[Binomial[n+m+l+k+1,2k](Binomial[n+l-k+1,2(l-k)]+
  Binomial[n+l-k,2(l-k)]),{k,0,l}],
  Element[n|m|l,Integers]&&n>=0&&m>=0&&l>=0]
Out[2]= 0
In[3]:= Fplus[n_][x_]:=((x+1)^n+(x-1)^n)/2
In[4]:= Simplify[Fplus[m+2n][x]+Fplus[m][x]x^(2n)-
  Sum[(Binomial[2n-k,k]+Binomial[2n-k-1,k-1])
  Fplus[m+k][x]x^k,{k,0,n}],
  Element[m|n,Integers]&&m>=0&&n>=0]
Out[4]= 0
In[5]:= Simplify[Fplus[m+2n+1][x]+Fplus[m][x]x^(2n+1)-
  Sum[Binomial[2n-k,k](Fplus[m+k+1][x]x^k+
  Fplus[m+k][x]x^(k+1)),{k,0,n}],
  Element[m|n,Integers]&&m>=0&&n>=0]
Out[5]= 0
```

□

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